Modeling the System for Infinite Impulse Response Digital Filter

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ABSTRACT: *In this work we have outlined the method for modelling the software realization of infinite impulse response (IIR) digital filter with one notch frequency. We observed that the introduced a configuration which has parallel nature with a pure delay in one path and third order all-pass phase corrector in another. The final filter completes all predefined specifications which is for arbitrary given maximal pass-band attenuation, band-stop boundary frequencies are symmetrical about notch frequency. With the help of illustrations, we have demonstrated the effectiveness, particularly by employing the Matlab system.*

Keywords: Notch Filter, Phase Corrector, All-pass Filters, Coupled All-pass, Phase Approximation

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I. Introduction

Digital narrow band-stop filters are widely used in applications where it is necessary to eliminate particular frequency component from input signal while the other frequency components need to remain untouched. Typical digital notch filters application areas include bio-medical engineering, seismology, speech processing, transmission of data through telephone channels, etc. The single-frequency interference can be removed by a notch filter tuned to that particular frequency. To remove unwanted signal components at several frequencies, it is possible to implement proposed filter in cascade configuration with adequately tuned their notch frequencies. In general, notch filters could be designed as recursive (Infinite Impulse Response-IIR) [1-2] or as nonrecursive (Finite Impulse Response-FIR) structures [3]. In case of FIR filters it is easy to achieve the exact linear phase thanks to existing transfer function coefficient's symmetry. To achieve the given magnitude specifications FIR filter demands significantly higher order compared to corresponding IIR filter [4].

The transfer function of narrow band-stop recursive filter could be determined by applying bilinear transform to the starting analog prototype filter of second order [3], [5]. In that case, the location of central frequency of the band-stop (notch frequency) as well as band-stop boundary frequencies at which attenuation reaches 3 dB value are the main parameters for the filter design [6]. However, the given specifications could not be achieved by this approach, i.e. boundary frequencies of the band-stop do not exhibit symmetry about predefined notch frequency. This deviation from an ideal solution is more evident in cases where the notch frequency is far from central frequency $\pi/2$, which corresponds to frequency $F_{sampling}/4$.

The proposed method is based on application of the third order phase corrector i.e. all-pass filter. The main goal is to achieve all given magnitude specifications. The realized notch frequency will be positioned exactly at predefined location, while at the same time cut-off edge frequencies are symmetrical regardless the value of the notch frequency. Practically, one wants full control of the magnitude characteristic at three frequencies. That is the main reason the third order polynomial is inevitable in all-pass transfer function. The coupled all-pass structure offers convenient way to solve the problem of filter design thanks to straightforward dependence of filter's magnitude and phase characteristic of corresponding all-pass sub-filter. Problem of design of resulting filter magnitude is easy to reformulate as the all-pass filter phase approximation problem.

The rest of the paper is structured as follows. In the Section II, the basic relations are derived according to which it is possible to obtain coefficients of the phase corrector transfer function. Comparison of the standard solution notch filter and notch filter obtained by the proposed procedure is done in Section III. Standard solution notch filter is realized by parallel connection of two all-pass filters with approximately constant phase in all pass-bands. The results of the simulation of designed filter performance are given in Section IV along with difference equations implemented in the Matlab® in order to obtain output of the notch filter.

2. Problem Definition

Specifications of notch filter magnitude characteristic precisely define the location of the notch frequency ω_n as well as band-stop width B_w. According to this available information it is possible to obtain band-stop lower ω_l and upper ω_r edge frequencies

$$\begin{aligned}\omega_l &= \omega_n - B_w / 2 \\ \omega_r &= \omega_n + B_w / 2\end{aligned} \tag{1}$$

based on the mentioned symmetry. Given parameters uniquely define maximal attenuation in both pass-bands a. Usually, for attenuation at cut-off frequencies the value of 3 dB is adopted. In this paper, the proposed method allows arbitrary positive value for maximal attenuation in pass-bands to be chosen. The transfer function of the narrow band-stop recursive filter is given with

$$H(z) = \frac{1}{2}(z^{-1} + \mathrm{P}(z)). \tag{2}$$

The transfer function of a stable phase corrector of the third order $P(z)$ is given with

$$\mathrm{P}(z) = z^{-3}\frac{D(z^{-1})}{D(z)}, \mathrm{D}(z) = 1 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3} \tag{3}$$

in parallel with first order delay line, as it is shown in Figure 1.

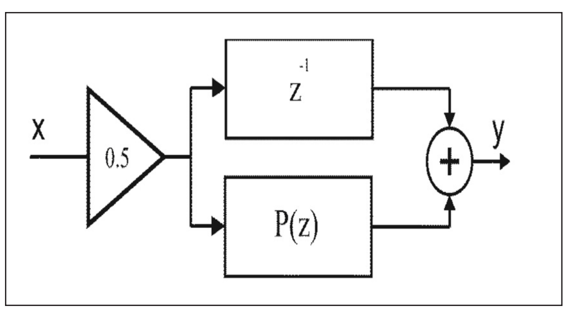

Figure 1. Coupled all-pass notch filter

By substituting Eq. (3), alongside with $z = exp(j\omega)$, into Eq. (2), after simple mathematical manipulations the magnitude characteristic of the notch filter could be obtained

$$|\mathrm{H}(e^{j\omega})| = \left|\frac{p_1 + (1+p_2)\cos\omega + p_3\cos 2\omega}{D(e^{j\omega})}\right|. \tag{4}$$

From the other hand, the following relation

$$p_1 + (1+p_2)\cos\omega + p_3\cos 2\omega = D_{\mathrm{Re}}(\omega)\cos\omega - D_{\mathrm{Im}}(\omega)\sin\omega \tag{5}$$

holds, where with $D_{Re}(\omega)$ and $D_{Im}(\omega)$ are represented real and imaginary part of the frequency response of the stable, minimal phase filter $D(z)$, respectively. Based on trigonometric identities

$$\cos(\arg D(e^{j\omega})) = \frac{D_{\mathrm{Re}}(\omega)}{|D(e^{j\omega})|},\ \sin(\arg D(e^{j\omega})) = \frac{D_{\mathrm{Im}}(\omega)}{|D(e^{j\omega})|} \tag{6}$$

Eq. (4) could be rewritten in more compact form

$$|H(e^{j\omega})| = |\cos(\omega + \varphi(\omega))| \tag{7}$$

where next notation is introduced

$$\varphi(\omega) = \arg D(e^{j\omega}) = -\arctan\frac{p_1\sin\omega + p_2\sin 2\omega + p_3\sin 3\omega}{1 + p_1\cos\omega + p_2\cos 2\omega + p_3\cos 3\omega}. \tag{8}$$

Taking into account the fact that function $\omega+\varphi(\omega)$ has value equal to zero for $\omega = 0$, and value π for $\omega = \pi$, coefficients p_1, p_2 and p_3 of the transfer function given with (2) need to be obtained such a way, to hold

$$\begin{gathered} |\omega + \varphi(\omega)| \le \varepsilon = \arccos 10^{-a/20}, \omega \in [0, \omega_l] \\ |\omega + \varphi(\omega) - \pi| \le \varepsilon, \omega \in [\omega_r, \pi] \end{gathered} \tag{9}$$

in both lower and upper pass-band.

According to the Eq. (4), one could conclude that amplification of the notch filter at frequency ω_n could be zero if the value of parameter p_1 is defined according to following equation

$$p_1 = -[p_3\cos 2\omega_n + (1+p_2)\cos\omega_n]. \tag{10}$$

Incorporating Eq. (10) in Eq. (8) after some mathematical manipulations, expression

$$\begin{aligned} & p_2[\sin(\varphi(\omega)+\omega)(\cos\omega - \cos\omega_n) + \\ & \cos(\varphi(\omega)+\omega)\sin\omega] + p_3[\sin(\varphi(\omega)+\omega)\cdot \\ & (\cos 2\omega - \cos 2\omega_n) + \cos(\varphi(\omega)+\omega)\sin 2\omega] \\ & = \cos\omega_n \sin(\varphi(\omega)+\omega) - \sin\varphi(\omega) \end{aligned} \tag{11}$$

is obtained, which give mutual dependence among parameters p_2, p_3 and the phase characteristic $\varphi(\omega)$. In order to adequately obtain the values of unknown parameters p_2 and p_3 (what would uniquely define the value of parameter p_1 according to Eq. (10)), it is necessary to know exact values of phase characteristic $\varphi(\omega)$ at $M \geq 2$ frequencies. Based on that data, a system of M equations could be formed with two unknowns. For $M = 2$, the system of equations has the unique solution, while for $M > 2$ one is forced to find least-square (*LSE*) solution. To remind a reader that magnitude of resulting couple all-pass filter and phase of all-pass sub-filter are mutually related. The specifications adopted for cut-off could be written in the form

$$\begin{aligned} \omega_l + \varphi(\omega_l) &= \varepsilon \\ \omega_r + \varphi(\omega_r) &= \pi - \varepsilon. \end{aligned} \tag{12}$$

Eq. (12) define the precise location of two points at phase characteristic $\varphi(\omega)$ with coordinates $(\omega_l, \varepsilon-\omega_l)$ and $(\omega_r, \pi-\varepsilon-\omega_r)$. The system of equations generated by using Eq. (11) applied to that two particular points leads to the unique solution for parameters p_2 and p_3:

$$\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sin\varepsilon\cos\omega_n - \sin(\varepsilon-\omega_l) \\ \sin\varepsilon\cos\omega_n - \sin(\varepsilon+\omega_r) \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} a_{11} &= \sin(\varepsilon+\omega_l) - \sin\varepsilon\cos\omega_n \\ a_{12} &= \sin\left(\varepsilon+2\omega_l\right) - \sin\varepsilon\cos 2\omega_n \\ a_{21} &= \sin(\varepsilon-\omega_r) - \sin\varepsilon\cos\omega_n \\ a_{22} &= \sin\left(\varepsilon-2\omega_r\right) - \sin\varepsilon\cos 2\omega_n. \end{aligned} \tag{14}$$

The obtained parameters not necessarily fulfill the conditions given by Eq. (9). From the other hand, using the software package Wolfram Mathematica, it was numerically justified that filter which coefficients are obtained by Eqs. (13) and (10), would met specifications given by Eq. (9) in a wide range of maximal allowed attenuation i.e. $a\in[0.001, 3]$ dB, for arbitrary notch frequency location from a set $\omega_n\in[0.1\pi, 0.9\pi]$ rad and with band-stop width $B_w\in[0.001\pi, 0.1\pi]$ rad. In other words, it holds

$$\min \frac{20\log_{10}|\cos(\omega+\varphi(\omega, B_w, \omega_n, a))|}{a} > -1, \\ \omega\in[0,\omega_l)\cup(\omega_r,\pi]. \tag{15}$$

3. Comparison With Standard Notch Filter

The transfer function of standard digital notch IIR filter [1], based on application of second order phase corrector is given with:

$$H_c(z) = \frac{1}{2}\left[1 + \frac{k_2 + k_1(1+k_2)z^{-1} + z^{-2}}{1 + k_1(1+k_2)z^{-1} + k_2 z^{-2}}\right] \tag{16}$$

where parameters k_1 and k_2 are given with

$$k_1 = -\cos\omega_n, k_2 = \frac{1-\tan B_w/2}{1+\tan B_w/2} \tag{17}$$

The maximal band-pass attenuation has value of 3 dB.

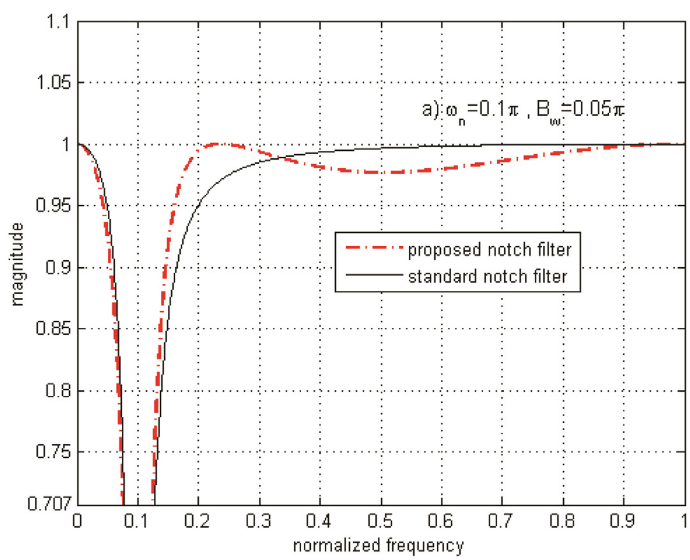

Figure 2. Magnitude of proposed and standard notch filter with notch frequency $\omega_n = 0.1\pi$ and $B_w = 0.05\pi$

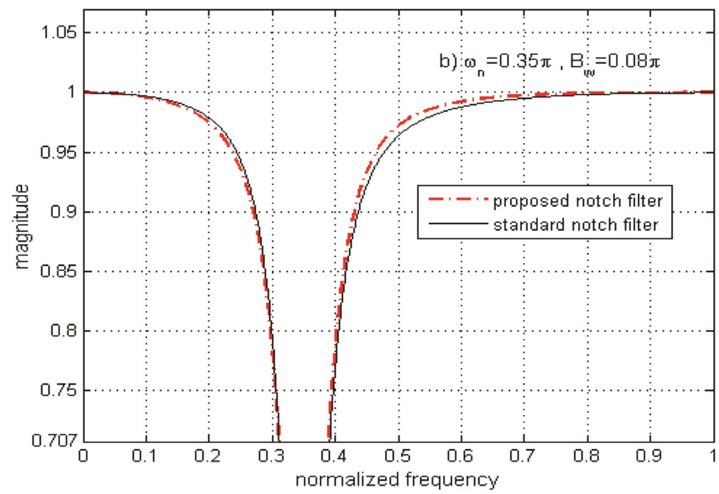

Figure 3. Magnitude of proposed and standard notch filter with notch frequency ω_n = 0.35π and B_w=0.08π

Magnitude characteristics of proposed notch filters alongside with standard solution, for different notch frequencies are presented in the following three figures. Fig. 2 corresponds to filters with notch frequency ω_n =0.1π with band-stop width B_w = 0.05π. Figs. 3 and 4 show results for filters with specifications ω_n = 0.35π, B_w=0.08π and ω_n =0.8π, B_w=0.1π, respectively. In order to facilitate comparison, all presented filters have the same attenuation of 3 dB at boundary frequencies.

According to the obtained results, displayed in given three figures, one can conclude that improvement in magnitude characteristic is more significant if the notch frequency is close to *π*, or even better results for notch frequency in vicinity of zero. That fact provides opportunity to increase sampling rate in order to improve filter efficiency in a simple way. The enhancement is negligible for filters with notch frequency which is located at central zone (about *π*/2). To remind the reader that this improvement is natural, behalf to slightly higher order of implemented filter. Standard notch filter solution involve the second order all-pass sub-filter in one of parallel branches, while the proposed filter apply third order all-pass sub-filter. One more unknown filter's coefficient offers the opportunity to expand the system of equations in order to fulfill the all given notch filter specifications.

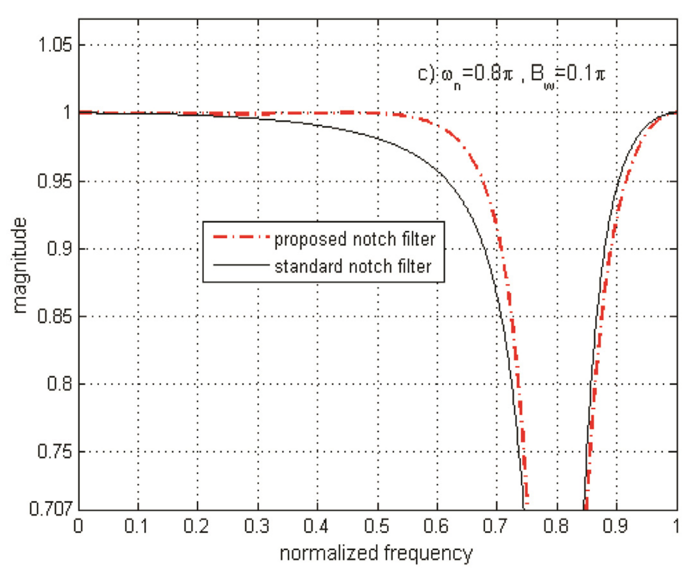

Figure 4. Magnitude of proposed and standard notch filter with notch frequency ω_n =0.8π and B_w= 0.1π

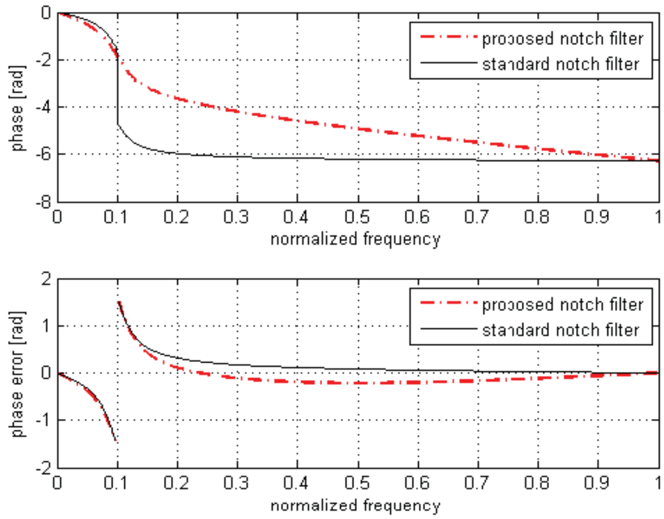

Figure 5. Phase and phase error of proposed and standard notch filter with notch frequency ω_n =0.1π and B_w=0.05π

Phases of the proposed and standard notch filter and corresponding phase errors are displayed in Fig.5. Standard filter's phase approximates constant phase in both pass-bands, while proposed filter's phase approximate piecewise linear phase. It is not possible to obtain filter with only one notch frequency with standard solution if order of all-pass sub-filter from one of parallel branches is higher than 2. Taking into account the fact that constant equal to one is in one path (with zero phase) choosing all-pass filter of order $2k$ for another path will lead to resulting filter with k notch frequencies. In other words, it is not possible to improve magnitude characteristic of standard filter with one notch frequency by choosing higher order all-pass sub-filter. On the other hand the proposed filter configuration does not have such constraint. Increasing of order of all-pass sub-filter has to be simply followed by increasing of delay line order. If difference in their orders remain to be equal to 2, resulting filter possess only one notch frequency. Choosing sub-filters of higher order give opportunity to reduce simultaneously the phase and magnitude error [8].

4. Software Realization and the Simulation Results

The Matlab® software package is used for design and realization of the proposed narrow band-stop filter. Sinusoidal noise with amplitude 0.2, at power-line frequency F_n = 50Hz, is superimposed on the electrocardiogram (ECG) signal $s[n]$ downloaded from the database MIT-BIH [7], as it is given in Eq. (18). All available signals in MIT-BIH database are recorded after digitalization using sampling frequency F_s = 360 Hz

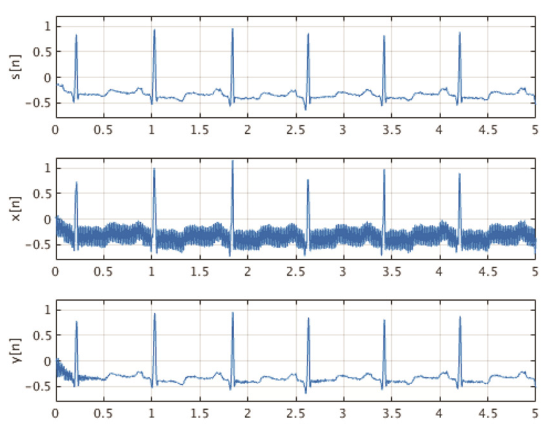

Figure 6. ECG signal $s[n]$, ECG signal with added sinusoidal noise $x[n]$ and filtered signal at output of notch filter $y[n]$

Specifications of the digital filter in z domain are: $\omega_n = 2\pi F_n / F_s = 5\pi/18$, while for band-stop width and maximal attenuation in pass-bands are adopted $B_w = 0.02\pi$ and $a = 1$dB, respectively. Software realization of the designed filter is based on linear difference equation derived according to Eq. (2)

$$y[n] = \frac{1}{2}(x[n-1] + w[n]), \; w[n] = x[n-3] + \sum_{i=1}^{3} p_i (x[n-3+i] - w[n-i]) \tag{19}$$

In Figure 6, ECG signal $s[n]$ which corresponds to 100th sample in MIT-BIH database is shown, together with corrupted version of this signal $x[n]$ after addition of sinusoidal noise. Filtered version of the signal at output of proposed notch filter $y[n]$ is also presented in Fig. 6. It is evident that designed notch filter successfully eliminated sinusoidal noise at powerline frequency from ECG signal. In the steady state, output signal $y[n]$ is noise free.

5. Conclusion

Design and software realization of IIR digital filter with one notch frequency is presented in this paper. The proposed filter is realized as parallel structure with pure delay in one path and all-pass sub-filter in another path. The existing methods for design of notch filter do not deliver solution with cut-off frequency symmetry about notch frequency. The proposed filter fulfills all predefined specifications i.e. for arbitrary given maximal pass-band attenuation, band-stop boundary frequencies are symmetrical about notch frequency. The efficiency of the presented method is illustrated by filtering ECG input signal corrupted with sinusoidal noise at powerline frequency. The proposed filter has approximately linear phase in pass-bands and minimal order. The phase and magnitude error of one notch filter could be further reduced by increasing simultaneously the order of delay element and all-pass sub-filter.

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References

[1] Joshi, Y. V., Dutta Roy, S.C. (1997). Design of IIR digital notch filters, *Circuits, Systems and Signal Processing*, 16 (4), p 415-427, 1997.

[2] Stancic, G., Nikolic, S. (2013). Design of digital recursive notch filter with linear phase characteristic, 11th International Conference on Telecommunications in Modern Satellite, *Cable and Broadcasting Services (TELSIKS)*, p 69-72, 2013.

[3] Roy, S. D., Kumar, B., Jain, B. (2001). FIR notch filter design: A review, *Facta Universitatis Series*, 14 (3), p 295-327.

[4] Aboutabikh, K., Haidar, I., Aboukerdah, N. (2016). Design and implementation of a digital FIR notch filter for the ECG signals using FPGA, *IJARCCE*, 5 (1), p 452-456, 2016.

[5] Hirano, K., Nishimura, S., Mitra, S. (1974). Design of Digital Notch Filters, *IEEE Transactions on Communications*, 22(7), *p* 964-970.

[6] Nikolic, S., Krsti, I., Stancic, G. (2018). Noniterative design of IIR multiple-notch filters with improved passband magnitude response, *Int. Journal of Circuit Theory and Applications*, 46 (12), *p* 2561-2567.

[7] Moody, G. B., Mark, R. G. (2001). The impact of the MIT-BIH Arrhythmia Database, *IEEE Eng in Med and Biol,* 20(3), p 45-50.

[8] Stancic, G., Nikolic, S. (2013). Digital linear phase notch filter design based on IIR all-pass filter application, *Digital Signal Processing*, 23(3), p1065-1069.