A Method for Optimization issue of Quasi-logarithmic Quantizers

Zoran Peric[1], Aleksandra Jovanovic[2], Milan Tancic[3]
Faculty of Electronic Engineering at University of Niš
Aleksandra Medvedeva 14
Niš 18000, Serbia
{zoran.peric@elfak.ni.ac.rs} {aleksandra.jovanovic@elfak.ni.ac.rs} {milan_tancic@yahoo.com}

ABSTRACT: *This work has addressed the optimization issue of quasi-logarithmic quantizer for memoryless Laplacian source. An iterative method for determination of compression factor* μ *and support limit* x_{max} *that gives the highest signal-to-quantization noise ratio SQNR is advocated. Further we have explained the optimization procedure. The experimental results support the suitability and success of this model.*

Keywords: Companding Quantizer, Quasi-logarithmic Compression Function, Laplacian Probability Density Function, Iterative Optimization Method

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1. Introduction

Nonuniform quantization has smaller decision intervals where the probability of input signal is high and larger intervals otherwise. To determine these decision intervals one of approaches is based on companding technique where an input signal is compressed, after that uniformly quantized and lastly expanded. This means that nonuniform quantizer can be implemented as a series connection of the compressor, the uniform quantizer and expandor, as shown in Figure 1. Companding quantizers can be optimal and logarithmic, and within the logarithmic there are A and μ- logarithmic quantizers. The companding concept is useful in analyzing nonuniform quantizers with a large number of levels [1].

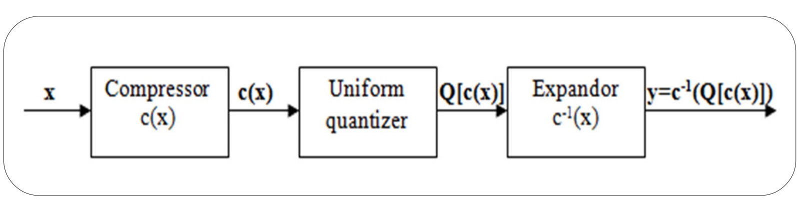

Figure 1. Companding quantizer

Logarithmic companding quantizers is mainly used when it is necessary to provide approximately constant signal-to-quantization nose (SQNR) in wide range of input signal variance, i.e., when it is necessary that the SQNR is not dependent on standard deviation. This is the case with non-stationary signals, which include the speech signal [1].

In this paper, emphasis is placed on designing quasi-logarithmic (μ- logarithmic) quantizer, whose compression factor μ significantly affects SQNR. The increase of the compression factor increases the range of variances in which the SQNR is approximately constant, but in the other hand reduces the value of SQNR. In order to provide robustness in a wide range of variance, in the ITU-T standard G.711 compression factor is 255 [1].

This paper explains determination of compression factor μ for which can be obtained the maximal value for the SQNR in a case when a memoryless Laplacian source is quantized. The obtained solution can be used in processing the speech signals, because these signals can be modeled using Laplacian probability density function [1]. The problem of determining the optimal compression factor with respect to SQNR was discussed in [2] too, where the first derivate of signal distortion in respect to μ was set to zero. Equation was directly solved by using iterative Muller's method [2].

The goal of this paper is to determine a simpler iterative method for compression factor optimization which gives results with same accuracy. In this paper, like in [2], we start from expression for total quantizer distortion, which we differentiate over compression factor μ and obtaining complex transcendental equation. We study the problem of solving the obtained transcendental equation and notice that it is possible to introduce a substitution which this transcendental equation transforms into an iterative quadratic equation whose solutions can be easily determined. We determine the solution, replace the introduced substitution and obtain simple iterative equation for the determination of the optimal value for compression factor μ. One of the parameters of iterative equation is support limit of quantizer and it must be correctly determined due to its influence on the SQNR [3], [4], [5], [6]. Wider support gives smaller overload distortion, but larger granular distortion and vice versa. In [5], for quasi-logarithmic quantizer a formula for determination of the quantizer support limit was derived. An application of this formula in [2] gave satisfactory results, so in this paper the same formula is used for determining the quantizer support limit. As the formula for support limit defining does not depend just on number of quantization levels N, but also on value of compression factor μ, this led us to iterative method whose iteration consists of two steps. At the first step, for the given support limit we determine new value for μ utilizing the equation derived in this paper, and then at the second step, for the obtained value of μ we determine support limit by the method given in [5]. Process is iteratively repeated until the absolute error between distortions in adjacent iterations is greater than pre-determined threshold. We show that this condition is satisfied in two iterations. We determine the optimal compression factor μ and the optimal support limit $x$max for the large numbers of levels N (128, 256, 512). The performances of quantizer for the number of levels $N = 256$, obtained by the proposed iterative method are compared with those performances from [2].

The paper is organized as follows. In Section 2 a description of companding quatizer for a Laplacian source is given and the performances and approximations that are introduced are presented. In Section 3 numerical results are shown and discussion of these results is given. In Section 4 the contributions of the paper are summarized.

2. Companding Quantizer for Laplacian Source

The process of quantization is the mapping $Q([x_{i-1}, x_i) \rightarrow y_i$, $i = 1,..., N$, where N is the number of quantizer levels, x_i, $i = 0,..., N$, are decision thresholds, and y_i, $i = 1,..., N$, are representation levels. When quantization is performed, an irreversible error due to rounding of the current value of input sample on representation level is made. This error is called quantization error. The average value of quantization error is defined as distortion. To calculate the distortion it is necessary to determine the actual values of samples and the representation levels at which these samples are rounded. Distortion is defined by expression [1]:

$$D = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - y_i)^2 \, p(x) dx \tag{1}$$

For a symmetric source densities with infinite support, distortion D is equal to the sum of granular and overload distortions $D = D_g + D_o$, which are defined as:

$$D_g = \sum_{i=2}^{N-1} \int_{x_{i-1}}^{x_i} (x - y_i)^2 p(x)dx \qquad (2)$$

$$D_o = 2 \int_{x_{N-1}}^{x_N} (x - y_N)^2 p(x)dx\,, \qquad (3)$$

where $x_{N-1} = x_{max}$ represents a quantizer support limit and $x_N = \infty$. In quasi-logarithmic quantization a compressor function used for signal compression is defined by:

$$c_\mu(x) = \frac{x_{\max}}{\ln(1+\mu)} \ln\left(1 + \mu \frac{|x|}{x_{\max}}\right) \operatorname{sgn}(x),\ |x| \le x_{\max}\,, \qquad (4)$$

where μ is compression factor [1]. The granular distortion (2) for companding quantizer can be defined by means of Benet's integral in following way [1], [2]:

$$D_g(Q_\mu) = \frac{x_{\max}^2}{3(N-2)^2} \int_{-x_{\max}}^{x_{\max}} \frac{p(x)}{\left[c'_\mu(x)\right]^2} dx \qquad (5)$$

where $p(x)$ is the probability density function of the signal. In this paper we consider Laplacian probability density function by which the speech signals can be modeled [1].

$$p(x) = \frac{1}{\sqrt{2}\sigma} \exp\left(-\frac{|x|\sqrt{2}}{\sigma}\right) \qquad (6)$$

By substituting (6) in (5) and (3)and using the approximation $y_N \approx x_{max}$ [2], [3], we can calculate the granular and overload distortion:

$$D_g(Q_\mu) = \frac{\ln^2(1+\mu)\sigma^2}{3N^2}\left[\frac{1}{\mu^2}\frac{x_{\max}^2}{\sigma^2} + \frac{x_{\max}}{\sigma}\frac{\sqrt{2}}{\mu} + 1\right] \qquad (7)$$

$$D_o(Q_\mu) = \sigma^2 \exp\left(-\frac{\sqrt{2}x_{\max}}{\sigma}\right) \qquad (8)$$

and after that, the total distortion $D(Q_\mu) = D_g(Q_\mu) + D_o(Q_\mu)$. As we design companding quantizer for unit varinace $\sigma^2 = 1$, the expression for total distortion becomes:

$$D(Q_\mu) = \frac{\ln^2(1+\mu)}{3N^2}\left(\frac{x_{\max}^2}{\mu^2} + \frac{\sqrt{2}x_{\max}}{\mu} + 1\right) + \exp\left(-\sqrt{2}x_{\max}\right) \qquad (9)$$

By equalizing the first derivate of distortion (9) in respect to μ with zero, we obtain the following equation:

$$\mu^2 + \sqrt{2}x_{\max}\left(1 - \frac{\mu+1}{\mu}\frac{\ln(\mu+1)}{2}\right)\mu + x_{\max}^2\left(\frac{\mu+1}{\mu}\ln(\mu+1)\right) = 0 \qquad (10)$$

In equation (10) we introduce the substitution:

$$t_0(\mu) = \frac{\mu+1}{\mu}\ln(\mu+1) \tag{11}$$

Which reduces transcendental equation (10) to iterative quadratic equation:

$$\left(\mu^{(i)}\right)^2 + \sqrt{2}x_{\max}\left(1 - \frac{t_0\left(\mu^{(i-1)}\right)}{2}\right)\mu^{(i)} + x_{\max}^2\left(1 - t_0\left(\mu^{(i-1)}\right)\right) = 0 \tag{12}$$

This iterative quadratic equation has two solutions:

$$\mu^{(i)} = \frac{\sqrt{2}x_{\max}}{4}\left[\left(t_0\left(\mu^{(i-1)}\right) - 2\right) \pm \sqrt{\left(t_0\left(\mu^{(i-1)}\right)\right)^2 + t_0\left(\mu^{(i-1)}\right) - 1}\right] \tag{13}$$

but we consider only solution that has physical sense, that is solution that gives positive values for compression factor.

Eq. (13) shows that the compression factor μ depends on the support limit x_{max}. We determine optimal support of quantizer utilizing method from [5]. The analysis conducted in [5] showed that for given N, the optimization of the support limit can be performed by means of following equation:

$$x_{\max} = \frac{1}{\sqrt{2}}\ln\left(\frac{3\mu N^2}{\ln^2(\mu+1)}\right). \tag{14}$$

In such way we obtain the system of two nonlinear equations (see Eqs. (13) and (14)) that can be solved iteratively. Actually, we replace $t_0(\mu^{(i-1)})$ with (11) in (13) and μ with $\mu^{(i)}$ in (14) and thus formulate a new iterative method for the determination of the optimal compression factor μ and the optimal support limit x_{max}, that consists from two steps:

$$\mu^{(i)} = \frac{\sqrt{2}x_{\max}^{(i-1)}}{4}\left[\left(\left(\frac{\mu^{(i-1)}+1}{\mu^{(i-1)}}\ln\left(\mu^{(i-1)}+1\right)\right) - 2\right) + \sqrt{\left(\frac{\mu^{(i-1)}+1}{\mu^{(i-1)}}\ln\left(\mu^{(i-1)}+1\right)\right)^2 + \left(\frac{\mu^{(i-1)}+1}{\mu^{(i-1)}}\ln\left(\mu^{(i-1)}+1\right)\right) - 1}\right] \tag{15}$$

$$x_{\max}^{(i)} = \frac{1}{\sqrt{2}}\ln\left(\frac{3\mu^{(i)}N^2}{\ln^2\left(\mu^{(i)}+1\right)}\right). \tag{16}$$

Quality of quantized signal is usually expressed through signal-to-quantization noise ratio which is defined as:

$$\text{SQNR}[dB] = 10\log_{10}\frac{\sigma^2}{D} \tag{17}$$

3. Iterative Method for the Optimal Compression Factor Determining and Numerical Results

In this chapter a detailed description of algorithm of the new iterative method for the optimization of quasi-logarithmic quantizer for Laplacian source is given. Optimization of compression factor μ and support limit x_{max} consists of following steps:

1. Set the iteration counter to zero: $i = 0$.

2. Set the initial value for the compression factor $\mu(i) = 128$, and the initial value for the support limit of quantizer $x_{max}(i)$ using expression (14), and after that, using expression (9) calculate the distortion $D(i)$.

3. Storage the values of compression factor, support limit and distortion before the calculation of new values: $\mu^{(i-1)} = \mu^{(i)}$, $x_{max}^{(i-1)} = x_{max}^{(i)}$, $D^{(i-1)} = D^{(i)}$.

4. Increase the iteration counter for 1: $i = i + 1$.

5. For given $x_{max}^{(i-1)}$, using the equation (15) calculate the compression factor $\mu^{(i)}$.

6. For given $\mu^{(i)}$, using the equation (16) calculate the support limit of quantizer $x_{max}^{(i)}$.

7. By using equation (9) determine distortion $D(i)$ and examine whether the difference which has distortion in adjacent iterations of algorithm is greater than a given threshold ε, or if $\delta = |D^{(i)} - D^{(i-1)}| > \varepsilon$ back to step 3, otherwise go to step 8.

8. Determine the optimal values of compression factor and support limit on following manner: $\mu^{opt} = \mu^{(i)}$ and $x_{max}^{opt} = x_{max}^{(i)}$.

We utilize the proposed algorithm to determine the optimal value of compression factor for N 128, 256 and 512. For threshold value for the distortion difference in two adjacent iterations ε we take10^{-4}, 10^{-5} or 10^{-6} depending on the distortion order. In Tables 1, 2 and 3 the iterative change of quantizer support limit x_{max}, compression factor μ, the distortion D, and the signal-to-quantization noise ratio SQNR is shown for N 128, 256 and 512, respectively. In Table 2 the SQNR when the compression factor is optimized utilizing the method from [2] is also shown. It can be seen that for all values of N, the optimal values of compression factor and support limit can be found in two iterations. It is evident that in comparison with the starting values of iterative method the SQNR sare increased for about 2dB. The result listed in tables also show that the optimal compression factor and support limit increase with N.

For the number of level $N = 256$, the optimal compression factor has value $\mu = 17.4769$, while the optimal value of support limit is $x_{max} = 9.15$. In paper [2] which is based on Muller's iterative method, the corresponding values are compression factor $\mu = 16.9227$ and support region threshold $x_{max} = 9.12$. The value of the SQNR obtained in the second iteration of new iterative method is SQNR = 40.4791 [dB], while in the 9^{th} iteration of Muller's iterative method this value is SQNR = 40.4835 [dB]. Comparing these results, we conclude that the SQNR obtained with the Muller's iterative method is slightly higher (for only 0.0044 dB) than that value obtained with new iterative method. It is evident that difference in SQNRs is very small, while the complexity of our iterative method is smaller than that in [2].

I	***M***	x_{max}	***D***	δ	***SQNR [dB]***
0	128	8.857	5.33×10^{-4}	–	32.729
1	25.535	8.272	3.50×10^{-4}	1.83×10^{-4}	35.558
2	15.02	8.1336	3.32×10^{-4}	1.8×10^{-5}	34.8731

Table 1. Compression factor, support limit, distortion and SQNR during iterative procedure for $n = 128$, $\varepsilon = 10^{-4}$.

i	***M***	x_{max}	***D***	δ	***SQNR [dB]***	***SQNR[2] [dB]***
0	128	9.84	1.34×10^{-4}	–	38.703	
1	28.269	9.28	9.32×10^{-5}	4.08×10^{-5}	40.305	40.483
2	17.476	9.15	8.95×10^{-5}	3.70×10^{-6}	40.479	

Table 2. Compression factor, support limit, distortion and SQNR during iterative procedure for $N = 256$, ε = 10-5

I	μ	x_{max}	D	δ	$SQNR$ [dB]
0	128	10.8231	3.4×10^{-5}	–	44.677
1	31.2025	10.2991	2.46×10^{-5}	9.4×10^{-6}	46.086
2	20.0058	10.1702	2.39×10^{-5}	7×10^{-7}	46.22

Table 3. Compression factor, support limit, distortion and SQNR during iterative procedure for $N = 512$, $\varepsilon = 10^{-6}$

In this paper we develop the method for the optimal compression factor and support limit determining. We obtain very good results in a few iterations what makes that this algorithm is a simple.

4. Conclusion

This paper proposes a new iterative method for optimization of μ companding quantizer for a Laplacian probability density function of unit variance. The new iterative method provides the optimal compression factor μ, as well as the optimal support limit $x$max that maximize signal-to-quantization noise ratio. Although the proposed iterative method solves complex system that consists of two nonlinear equations, it is a very simple. The iteration consists of two steps specified with (15) and (16), while the optimal values are determined in a few iterations, which points out the fast convergence of iterative method. The compression factor μ and the support limit x_{max} determined with this iterative method provides the signal-to-quantization noise ratio approximately equal with the signal-to-quantization noise ratio obtained using the iterative method for the compression factor optimization in [2], wherein the number of iterations is several times smaller.

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