Magnetic Scalar Potential of the Point Charge for Bi-isotropic Material

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ABSTRACT: *Using the image theorem we have prepared bi-isotropic material of Tellegen type for which the expressions for both electric and magnetic scalar potential of the point charge placed in air spherical hollow. It represents a contribution to the electromagnetic analysis of bi-isotropic materials and we have tested the design.*

Keywords: Electrostatics, Bi-isotropic Material, Image Theorem, Point Charge

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1. Introduction

Bi-isotropic materials have been incorporated in various electric devices and the research of specific problems is reported in a number of papers [1"5].In previous years, authors have been reporting solutions of different problems, which incorporate these mediums, on several occasions, using EEM (Equivalent Electrode Method), which indirectly uses Image theorem for obtaining Green's function [6"8], or by applying numerical methods such as FEM (Finite Element Method) [9]. Furthermore, image theorem is exploited in the analysis of bi-isotropic body placed in homogeneous electric field [10] or for field calculation of charged ring near biisotropic sphere[11]. Bi-isotropic materials can be described using the following constitutive relations [1, 2]

$$\mathbf{D} = \varepsilon \mathbf{E} + \xi \mathbf{H} \quad \text{and} \quad \boldsymbol{B} = \mu \boldsymbol{H} + \xi \boldsymbol{E} \tag{1}$$

where $\boldsymbol{D}$ represents displacement fieldvector, $\boldsymbol{E}$ is electric field vector, $\boldsymbol{H}$ is magnetizing field vector, $\boldsymbol{B}$ is magnetic field vector, while ε is permittivity of the material, ξ is permeability whereas x is the parameter which describes biisotropic characteristics of the material.

Such materials consist of elements that have permanent electrical and magnetic dipoles, parallel or antiparallel with others according to Tellegen. Thus, electric field in such material simultaneously orients both electrical and magnetic dipoles. Then as well, the magnetic field in such material orients both electric and magnetic dipoles at the same time.

Laplace's equation for the electric scalar potential is obtained starting from Maxwell's equations and constituent relation (1):

$$\Delta\varphi = -\rho_s / \varepsilon_e, \; \varepsilon_e = \varepsilon(1 - \xi^2 / \varepsilon\mu), \tag{2}$$

while Laplace's equation for the magnetic scalar potential of free electric charges in the bi-isotropic environment is:

$$\Delta\varphi_m = \xi\rho_s/(\varepsilon_e\mu), \tag{3}$$

where $\varepsilon\mu \neq \xi^2$ [12] and $\Delta(..)$ is the Laplacian function[13].

It is obligatory to integrate the Poisson's equations for the electric and magnetic scalar potential when determining the electric field in such environments, with respecting the existing boundary conditions. Then, field components are obtained using defining relations:

$$\boldsymbol{E} = -\mathrm{grad}\varphi \text{ and } \boldsymbol{H} = -\mathrm{grad}\varphi_m. \tag{4}$$

In this paper,both theoretical field and potential calculation of the electric and magnetic scalar potential for the case of a pointcharge, placed into the air spherical hollow, is presented. It represents a contribution to the electrostatics of bi-isotropic materials.

2. Point Charge in the Air Spherical Hollow in the Bi-isotropic Material- Problem Definition

If we set coordinate system such the charge is placed along z-axis while the center of air spherical hollow and the center of coordinate system coincide (Figure 1), both electric and magnetic scalar potential inside the sphere satisfies Poisson's equation whereas the potentials outside the sphere satisfies Laplace's equation. The system is axially symmetrical and the potentials do not depend on ψ - coordinate of the spherical coordinate system [13].

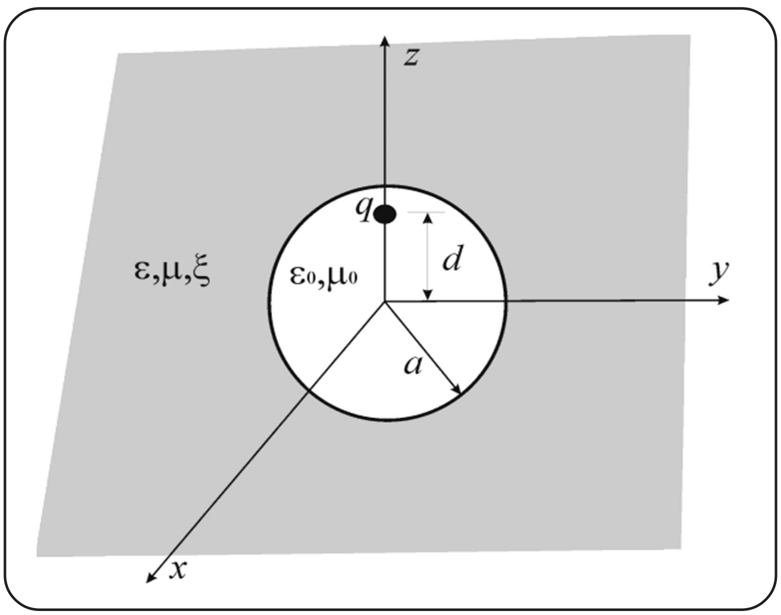

Figure 1. Point charge in the air spherical hollow which is placed in biisotropic material

The equation (2) for electric scalar potential is defined as:

$$\sin(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\varphi}{\partial\theta}\right)=\begin{cases}-\dfrac{q\delta(r-d)\delta(\theta)}{2\pi\varepsilon_0}, & r<a\\ 0, & r>a\end{cases} \tag{5}$$

whereas the equation for magnetic scalar potential is:

$$\sin(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\varphi}{\partial\theta}\right)=0,\ 0<r<\infty \tag{6}$$

In (5) δ is the Kronecker delta function. This is a function of two variables, usually just non-negative integers. The function is equal toone if the variables are equal ($r = d$), and equal to zero otherwise.

The solution can be represented in the form:

$$\varphi=\sum_{n=0}^{\infty}R_n(r)P_n(\cos(\theta)), \tag{7}$$

where P_n $(cos(\theta))$ are Legendre polynomials of the first kind[13].

If we substitute equation (7) in equation (6), using Legendre differential equation:

$$\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\mathrm{d}P_n}{\mathrm{d}\theta}\right)=-n(n+1)\sin(\theta), \tag{8}$$

where $P_n = P_n$ $(\cos(\theta))$, it is obtained:

$$\sum_{n=0}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R_n'\right)-n(n+1)R_n\right]P_n(\cos(\theta))\sin(\theta)=-\frac{q}{2\pi\varepsilon_0}\delta(r-d)\delta(\theta) \tag{9}$$

If we multiply the previous equation with $P_{n'}$ $(\cos(\theta))$, and integrate it with respect to θ, from 0 to π , we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R'_n\right)-n(n+1)R_n=-\frac{q}{4\pi\varepsilon_0}(2n+1)\delta(r-d), \tag{10}$$

taking into account that it is valid:

$$\int_0^{\pi}\sin(\theta)P_n(\cos\theta)P_{n'}(\cos(\theta))\,\mathrm{d}\theta=\begin{cases}0,\ n\neq n'\\ \dfrac{2}{2n+1},\ n=n'\end{cases} \tag{11}$$

and

$$P_n(1)=1$$

The equation (8) can be solved using the variation of constant method:

$$R_n(r)=\begin{cases}\mathrm{C}_nr^n+\mathrm{D}_n/r^{n+1}, r\leq d\\ \left(\mathrm{C}_n-\dfrac{q}{4\pi\varepsilon_0d^{n+1}}\right)r^n+\left(\mathrm{D}_n+\dfrac{qd^n}{4\pi\varepsilon_0}\right)\dfrac{1}{r^{n+1}}, d\leq r\leq a.\\ \mathrm{A}_nr^n+\dfrac{\mathrm{B}_n}{r^{n+1}}, a\leq r\end{cases} \tag{12}$$

Because the charge point is in the air, the equation (3) for magnetic scalar potential is:

$$\sin(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_m}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\varphi_m}{\partial\theta}\right)=0, \tag{13}$$

Analogously to the previous consideration, the solution for Eq (13) can be represented in the form:

$$\varphi_m = \sum_{n=0}^{\infty} R_{1n}(r) P_n(\cos(\theta)), \tag{14}$$

where function $R_{1n}(r)$is:

$$R_{1n}(r) = \begin{cases} C_{1n}\, r^n + D_{1n} / r^{n+1}, r \le a \\ A_{1n} r^n + \dfrac{B_{1n}}{r^{n+1}}, a \le r \end{cases} \tag{15}$$

As the electric scalar potential φ in (12) has a finite value in the center of the sphere, as well as in the infinity, it has to be valid:

$$D_n = 0 \text{ and } A_n = 0 \tag{16}$$

Similarly, magnetic scalar potential φ_m has a finite value in the center of the sphere and in the infinity, thus:

$$A_{1n} = 0 \text{ and } D_{1n} = 0 \tag{17}$$

in (15).

The remaining constants can be determined from the boundary conditions for the continuity of the electric scalar potential:

$$\varphi(r = a - 0) = \varphi(r = a + 0), \tag{18}$$

$$\varphi_m(r = a - 0) = \varphi_m(r = a + 0), \tag{19}$$

$$D_\mathrm{n}(r = a - 0) = D_\mathrm{n}(r = a + 0), \tag{20}$$

$$B_\mathrm{n}(r = a - 0) = B_\mathrm{n}(r = a + 0). \tag{21}$$

From the condition of electric scalar potential, φ, continuity, it is:

$$\frac{B_n}{a^{n+1}} = \left(C_n - \frac{q}{4\pi\varepsilon_0 d^{n+1}} \right) a^n + \frac{q d^n}{4\pi\varepsilon_0} \frac{1}{a^{n+1}}, \tag{22}$$

while from the magnetic scalar potential (φ_m) continuity condition:

$$C_{1n}\, a^n = \frac{B_{1n}}{a^{n+1}} \tag{23}$$

Furthermore, from the continuity condition of D_n on the boundary surface, where D_n is the normal component of vector $\boldsymbol{D}$, it is obtained:

$$\begin{aligned} &-\varepsilon(n+1)\frac{B_n}{a^{n+2}} - \xi(n+1)\frac{B_{1n}}{a^{n+2}} = \varepsilon_0 n a^{n-1}\left(C_n - \frac{q}{4\pi\varepsilon_0 d^{n+1}} \right) - \\ &\varepsilon_0(n+1)\left(\frac{q d^n}{4\pi\varepsilon_0} \right)\frac{1}{a^{n+2}}. \end{aligned} \tag{24}$$

Next, from the condition of vector B_n continuity on the boundary surface, where B_n is the normal component of vector $\boldsymbol{B}$, it can be obtained:

$$-\mu(n+1)\frac{B_{1n}}{a^{n+2}}-\xi(n+1)\frac{B_n}{a^{n+2}}=\mu_0 n a^{n-1} C_{1n} \tag{25}$$

In the end, it is obtained:

$$\begin{aligned} C_n &= \frac{q}{4\pi\varepsilon_0 d^{n+1}}+ \\ &\frac{qd^n}{4\pi\varepsilon_0 a^{2n+1}}\frac{(n+1)\left\{(\varepsilon_0-\varepsilon)[\mu(n+1)+\mu_0 n]+\xi^2(n+1)^2\right\}}{\left\{[\varepsilon(n+1)+\varepsilon_0 n][\mu(n+1)+\mu_0 n]-\xi^2(n+1)^2\right\}} \end{aligned} \tag{26}$$

$$B_n=\frac{qd^n}{4\pi}\frac{(2n+1)[\mu(n+1)+\mu_0 n]}{[\varepsilon(n+1)+\varepsilon_0 n]\left\{[\mu(n+1)+\mu_0 n]-\xi^2(n+1)^2\right\}} \tag{27}$$

$$B_{1n}=-\xi\frac{qd^n}{4\pi}\frac{(2n+1)(n+1)}{[\varepsilon(n+1)+\varepsilon_0 n]\left\{[\mu(n+1)+\mu_0 n]-\xi^2(n+1)^2\right\}} \tag{28}$$

$$C_{1n}=-\xi\frac{qd^n}{4\pi a^{2n+1}}\frac{(2n+1)(n+1)}{[\varepsilon(n+1)+\varepsilon_0 n]\left\{[\mu(n+1)+\mu_0 n]-\xi^2(n+1)^2\right\}} \tag{29}$$

3. Numerical Results

It is of interest to consider parameters which appear in the constants C_n, B_n, C_{1n}, B_{1n} and which depend on $\xi^2/\varepsilon_0\mu_0$. Let's consider the case $\varepsilon = 2\varepsilon_0$ and $\mu = \mu_0$. Furthermore, let's denoted parameters as:

$$p(\varepsilon/\varepsilon_0,\mu/\mu_0,n)=\frac{(n+1)\left\{\left(1-\frac{\varepsilon}{\varepsilon_0}\right)\left[(n+1)\frac{\mu}{\mu_0}+n\right]+\frac{\xi^2}{\varepsilon_0\mu_0}(n+1)^2\right\}}{\left[\frac{\varepsilon}{\varepsilon_0}(n+1)+n\right]\left[\frac{\mu}{\mu_0}(n+1)+n\right]-\frac{\xi^2}{\varepsilon_0\mu_0}(n+1)^2} \tag{30}$$

$$p_1(\varepsilon/\varepsilon_0,\mu/\mu_0,n)=\frac{(2n+1)(n+1)}{\left[\frac{\varepsilon}{\varepsilon_0}(n+1)+n\right]\left[\frac{\mu}{\mu_0}(n+1)+n\right]-\frac{\xi^2}{\varepsilon_0\mu_0}(n+1)^2} \tag{31}$$

$$\begin{aligned} \alpha(\varepsilon/\varepsilon_0,\mu/\mu_0,n) &= \left[\frac{\varepsilon}{\varepsilon_0}(n+1)+n\right]\left[\frac{\mu}{\mu_0}(n+1)+n\right] \\ &-\frac{\xi^2}{\varepsilon_0\mu_0}(n+1)^2 \end{aligned} \tag{32}$$

and let us show their graphical dependence on bi-isotropy parameter $\xi^2/(\varepsilon_0\mu_0)$, when n is parameter.

From Figure 2 and Figure 3 it can be seen that parameters p and p_1 increase by increasing the bi-isotropy parameter ξ, whereas parameter α decreases by increasing bi-isotropy parameter ξ. In these three figures, n is the parameter.

5. Conclusion

In this paper, we have derived electric and magnetic scalar potential of point charge which is placed in air spherical hollow, which exists inside bi-isotropic medium of Tellegen type. The expressions are obtained by solving Legendre differential equations for

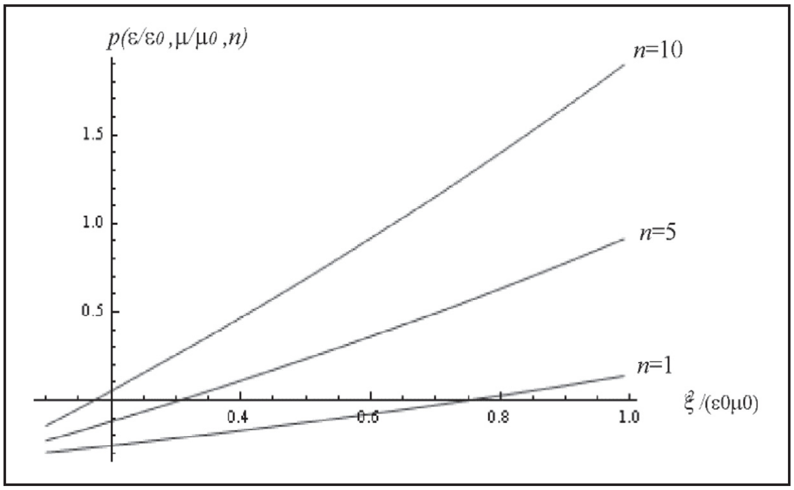

Figure 2. Dependence p ($\varepsilon / \varepsilon_0$, μ/μ_0, n) onbi-isotropy parameter

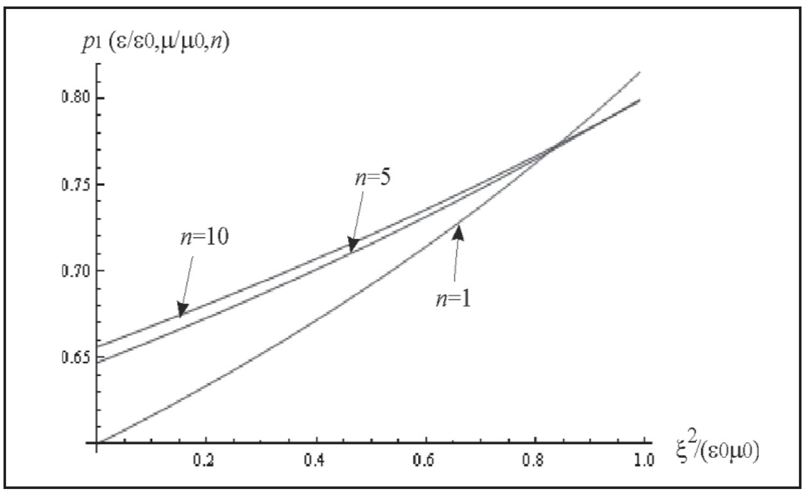

Figure 3. Dependence p_1 ($\varepsilon / \varepsilon_0$, μ/μ_0, n) onbi-isotropy parameter

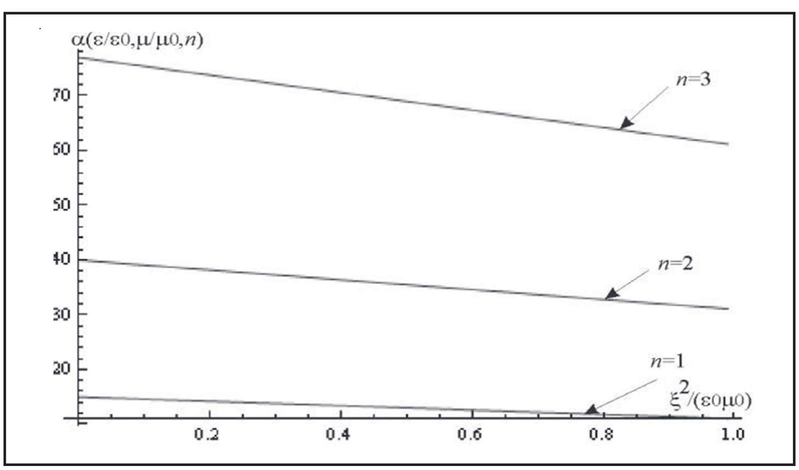

Figure 4. Dependence α ($\varepsilon / \varepsilon_0$, μ/μ_0, n) onbi-isotropy parameter ξ

electric and magnetic scalar potential and by satisfying boundary conditions for both potential continuity on boundary surfaces as well as continuity of normal components of vectors ***D*** and ***B***.

As a consequence of specific constitutive relations for biisotropic mediums, constants C_n, B_n, C_{1n}, B_{1n} depend on biisotropy parameter, which characterizes the bi-isotropic properties of the surrounding environment outside air spherical hollow.

This further influences both the electrical and magnetic scalar potential, as well as the fields that exist in the vicinity of the point charge, which are obtained by equation (4).In addition, the presence of Tellegen's bi-isotropic environment, although the point charge is placed in the air hollow, leads to the appearance of a magnetic field. When $\xi = 0$, expressions are reduced to those that are derived [13] and magnetic field does not exist.

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