Vilekin's Additional Theorem for Legendre Functions and Rotation-based Measurements

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ABSTRACT: *In this work, we applied Vilenkin's additional theorem among the circular apertures to measure the coupling calculations. We measure the vector-Legendre transformations of basis and test functions by keeping the domain over the aperture at an arbitrary level. With the help of Euler's formula for rotating the coordinate system, we used the classical approach for the numerical measurements, which is more time intensive. With Vilekin's additional theorem for Legendre functions, we increased the rotation-based measurements.*

Keywords: Spherical Antenna, Circular Aperture, Vilenkin's Additional Theorem, Method of Moments

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1. Introduction

An array of aperture antennas on the surface of a sphere is of importance because such an array provides wide hemispherical scan coverage with low grating lobe levels.

The mutual coupling of aperture antenna array is determined in this article by using the Vilenkin's theorem instead of the classic approach to calculations such attachment significantly shortens the calculation time. The computerized routine used is verified by using data from the available literature as well as by using measurements on a derived laboratory model.

2. Method of Analysis

The problem of determining the mutual coupling using the moment method (MoM) in spectral domain for an array circular apertures placed at a spherical ground plane is the problem of calculation equivalent magnetic current placed at different apertures (openings of waveguide Figure 1). The mutual admittance is given by:

$$\mathbf{Y}_{ij} = -\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} 2\pi S(n,m) r_s^2 \widetilde{\mathbf{M}}_i^{\mathrm{T}}(\mathrm{r}, n, -m) \overline{\overline{\widetilde{\mathbf{G}}}}(n, m, r_s | r_s) \widetilde{\mathbf{M}}_j(\mathrm{r}, n, m), \tag{1}$$

where $\overline{\overline{\widetilde{\mathbf{G}}}}(n,m,r_s|r_s)$ is a spectral domain dyadic Green's function for grounded spherical surface and $\overline{\overline{\mathbf{L}}}(n,m,\theta)$ is the kernel of the vector-Legendre transformation. $\widetilde{\mathbf{M}}_{\mathbf{j}}(r,n,m)$ and $\widetilde{\mathbf{M}}_{\mathbf{i}}^{\mathbf{T}}(r,n,m)$ are the equivalent magnetic current and the transposed equivalent magnetic current both placed at the i-th and j-th opening of waveguide. This basic/test functions are located at different apertures.

The appropriate spectral-domain Green's function of a multilayer spherical structure is calculated using the G1DMULT algorithm [2].

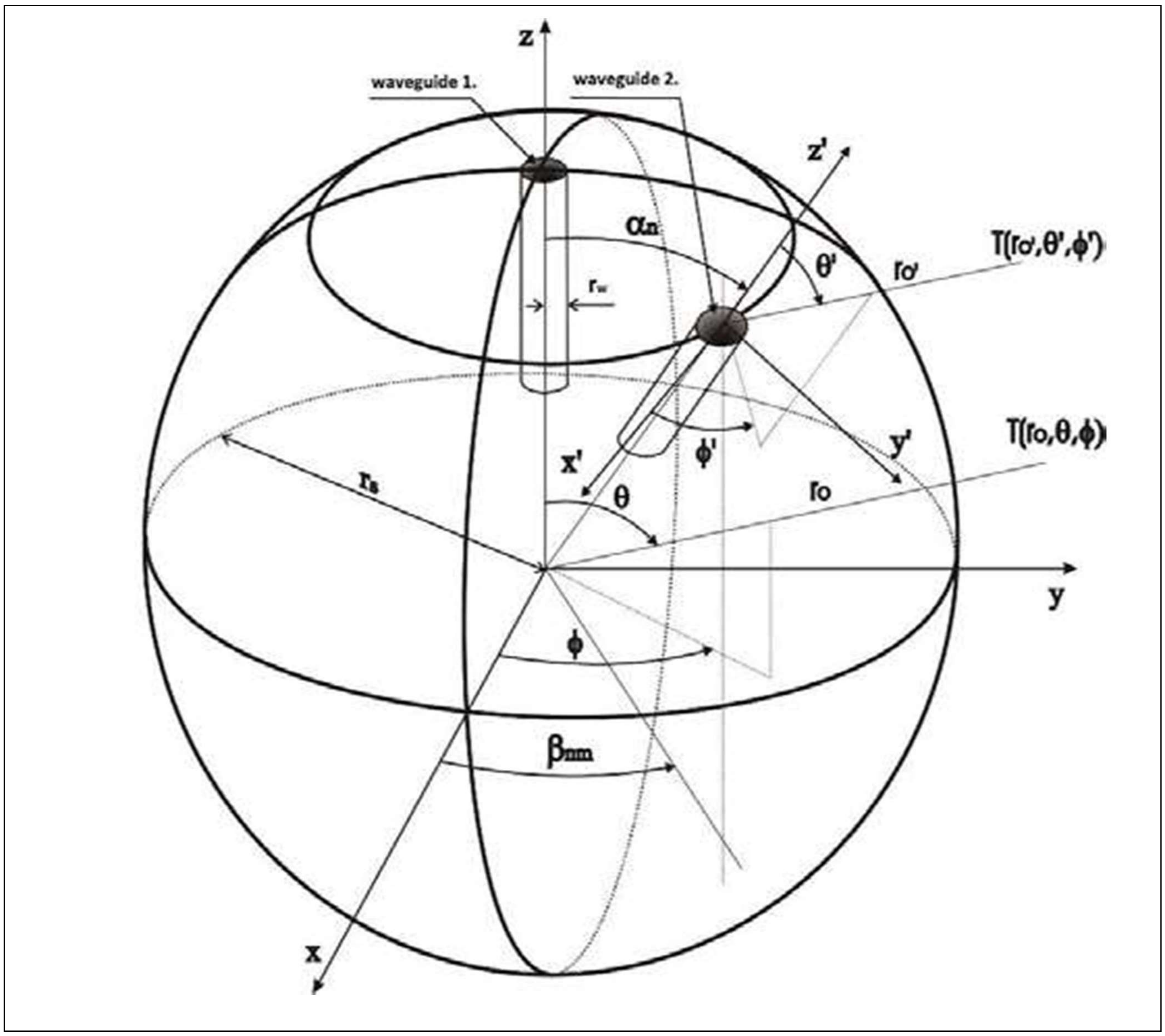

Figure 1. Circular waveguides (aperture antennas) on a spherical surface – spherical geometry

It is easy to calculate the vector-Legendre transformation of the equivalent current of the waveguide opening located on the north pole ($\alpha_n = 0$, $\beta_{nm} = 0$). However, when calculating the mutual coupling between the apertures, it is necessary to determine the vector-Legendre transformation of the base and test functions with the domain on the wavegudie aperture of the arbitrary position in the sphere. One way is to numerically determine the required expressions using formulas that link global and local coordinates. The transformation process is based on the vector-Legendre transformation of base and test functions located on the displaced waveguide:

$$\widetilde{\mathbf{M}}_{\mathbf{j}}(r,n,m) = \begin{bmatrix} 0 \\ \widetilde{M}_\theta \\ \widetilde{M}_\phi \end{bmatrix} = \frac{1}{2\pi\, S(n,m)} \iint_{wav} \widetilde{\overline{\mathbf{L}}}(n,m,\theta) \cdot \mathbf{M}_{\mathbf{j}}(r,\theta',\phi') \cdot \sin\theta' \; e^{-jm\phi} \; d\theta' \, d\phi' \tag{2}$$

where θ and ϕ are coordinates in the global coordinate system, and θ' and φ in the local coordinate system. It is important to know that both matrices, L-matrices and basis functions M_i are written in relation to the basis $(\hat{e}_\theta, \hat{e}_\phi, \hat{e}_r)$, which is the basis of the global coordinate system. The connection between local and global coordinates is given by the following equations:

$$\cos\theta = -\sin\alpha_n \sin\theta' \cos\phi + \cos\alpha_n \cos\theta' \tag{3a}$$

$$\cot\phi = \frac{\cos\alpha_n \sin\theta' \cos\phi' + \sin\alpha_n \cos\theta'}{\sin\theta' \sin\phi'} \tag{3b}$$

α_n and β_{nm} are the θ and ϕ-coordinates of the center of each waveguide in the global system.

This approach is very time consuming since for each basis/tes function one needs to calculate a double-integral of rapidly varying function.

A much more efficient and faster algorithm is the one in which ([6] and [7]) we have the following relationship between the vector Legendre transformations of the basis/test function with domain on the central apertures ($\theta = 0$) and on the aperture whose center has coordinate $\theta = \alpha_n$, $\phi = \beta_n = 0$. Equivalent magnetic density of the current at the aperture in the spatial coordinates is defined by the equation:

$$\mathbf{M}_k(r_{wav.aperture},\theta,\phi) = \nabla\times(\hat{r}\, r_{wav.aperture} M_k^A(r_{wav.aperture},\theta,\phi)) + \nabla\cdot(r_{wav.aperture} M_k^\psi(r_{wav.aperture},\theta,\phi)) \tag{4}$$

In the spherical coordinate system of equation (4) we can write:

$$\mathbf{M}_k(r_{wav.aperture},\theta,\phi) = \hat{\theta}\frac{1}{\sin\theta}\frac{\partial M_k^A}{\partial\phi} - \hat{\phi}\frac{\partial M_k^A}{\partial\theta} + \hat{\theta}\frac{\partial M_k^\psi}{\partial\theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial M_k^\psi}{\partial\phi} \tag{5}$$

Equations (4) and (5) are described with two functions M_k^A and M_k^ψ.

In addition, base and test functions can be written by using direct inverse vector Legendre transformation as:

$$\begin{aligned}\mathbf{M}_k(r,\theta,\phi) = \begin{bmatrix} 0 \\ M_\theta \\ M_\phi \end{bmatrix} = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty} e^{j\omega t}\Bigg\{&\hat{\theta}\left[\frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta} M_{k\theta}(r_{wav.aperture},n,m) - \frac{jmP_n^{|m|}(\cos\theta)}{\sin\theta} M_{k\phi}(r_{wav.aperture},n,m)\right] \\ &+\hat{\phi}\left[\frac{jmP_n^{|m|}(\cos\theta)}{\sin\theta} M_{k\theta}(r_{wav.aperture},n,m) + \frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta} M_{k\phi}(r_{wav.aperture},n,m)\right]\Bigg\}\end{aligned} \tag{6}$$

By comparing the relation (5) and (6), it is apparent that the functions M_k^A and M_k^ψ have the following form

$$M_k^A(r_{wav.aperture},n,m) = -M_{k\phi}(r_{wav.aperture},n,m)\,P_n^{|m|}(\cos\theta)\,e^{jm\phi},\; M_k^\psi(r_{wav.aperture},n,m) = M_{k\theta}(r_{wav.aperture},n,m)\,P_n^{|m|}(\cos\theta)\,e^{jm\phi} \quad (7)$$

Partial derivations are:

$$\frac{\partial M_k^A}{\partial\phi} = -jm\,M_{k\phi}(r_{wav.aperture},n,m)\,P_n^{|m|}(\cos\theta)\,e^{jm\phi} \quad (8a)$$

$$\frac{\partial M_k^A}{\partial\theta} = -M_{k\phi}(r_{wav.aperture},n,m)\frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta}e^{jm\phi} \quad (8b)$$

$$\frac{\partial M_k^\psi}{\partial\phi} = jm\,M_{k\theta}(r_{wav.aperture},n,m)\,P_n^{|m|}(\cos\theta)\,e^{jm\phi} \quad (8c)$$

$$\frac{\partial M_k^\psi}{\partial\theta} = M_{k\theta}(r_{wav.aperture},n,m)\frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta}e^{jm\phi} \quad (8d)$$

Furthermore, theta and phi components of equivalent currents can be written by the equations:

$$M_\theta = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}e^{j\omega t}\left\{\left[\frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta}M_{k\theta}(r_{wav.aperture},n,m) - \frac{jmP_n^{|m|}(\cos\theta)}{\sin\theta}M_{k\phi}(r_{wav.aperture},n,m)\right]\right\}e^{jm\phi} \quad (9)$$

$$M_\phi = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}e^{j\omega t}\left\{\left[\frac{jmP_n^{|m|}(\cos\theta)}{\sin\theta}M_{k\theta}(r_{wav.aperture},n,m) + \frac{\partial P_n^{|m|}(\cos\theta)}{\partial\theta}M_{k\phi}(r_{wav.aperture},n,m)\right]\right\}e^{jm\phi} \quad (10)$$

Or shorter written:

$$M_\theta = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}e^{j\omega t}\left\{\frac{1}{\sin\theta}\frac{\partial M_k^A}{\partial\phi} + \frac{\partial M_k^\psi}{\partial\theta}\right\} \quad (11)$$

$$M_\phi = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}e^{j\omega t}\left\{-\frac{\partial M_k^A}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial M_k^\psi}{\partial\phi}\right\} \quad (12)$$

The selected view is similar to the display of the electric field via vector and scalar potentials. It is also important to note that these relationships do not depend on the coordinate system, ie they are valid in the global and local coordinate system. It is possible to connect the equivalent currents M_k^A and M_k^ψ and in different coordinate systems using the following rule of Legendre functions:

$$e^{-jm\phi}P_n^{|m|}(\cos\theta) = j^m\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2}\sum_{k=-n}^{n}j^{-k}\left[\frac{(n-k)!}{(n+k)!}\right]^{1/2}P_{m,k}^n(\cos\theta_{12})P_n^{|k|}(\cos\theta')e^{-jk\phi'} \quad (13)$$

$$e^{jm\phi}P_n^{|m|}(\cos\theta) = \sum_{m=-n}^{n}(-j)^{|m|-|k|}\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}\sqrt{\frac{(n-|k|)!}{(n+|k|)!}}P_{|m|,sign(m)\cdot k}^n(\cos\theta_{12})P_n^{|k|}(\cos\theta')e^{jk\phi'} \quad (14)$$

The function , $P^{n}_{m,k}(\cos\theta_{12})$ is defined in [1], and θ_{12} is the angle between the global and the local coordinate system (it is important to note the following: local coordinates have a dash mark on the exponent's site, and the ϕ coordinate of the center of the shifted aperture is equal to zero).

By connecting the relation (7) and (14), the terms for functions M^{A}_{k} and M^{ψ}_{k} with the domain on the displaced wavelength can be determined:

$$M^{A}_{k}(r_{wav.aperture},n,k) = -P^{|k|}_{n}(\cos\theta')\, e^{jk\phi'} \sum_{m=-n}^{n}(-j)^{|m|-|k|}\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}\sqrt{\frac{(n-|k|)!}{(n+|k|)!}}P^{n}_{|m|,sign(m)\cdot k}(\cos\theta_{12})M_{k\phi}(r_{wav.aperture},n,m),$$

$$M^{\psi}_{k}(r_{wav.aperture},n,k) = P^{|k|}_{n}(\cos\theta')\, e^{jk\phi'} \sum_{m=-n}^{n}(-j)^{|m|-|k|}\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}\sqrt{\frac{(n-|k|)!}{(n+|k|)!}}P^{n}_{|m|,sign(m)\cdot k}(\cos\theta_{12})M_{k\theta}(r_{wav.aperture},n,m).$$

(15) and (16)

Furthermore, by comparing the relation (15) and (16) with the reaction (7), the following is obtained:

$$M^{A}_{k}(r_{wav.aperture},n,k) = -P^{|k|}_{n}(\cos\theta')\, e^{jk\phi'}M_{k\phi_S}(r_{wav.aperture},n,k), \quad (17)$$

$$M^{\psi}_{k}(r_{wav.aperture},n,k) = P^{|k|}_{n}(\cos\theta')\, e^{jk\phi'}M_{k\theta_S}(r_{wav.aperture},n,k). \quad (18)$$

Since we are interested in the theta and phi components of the equivalent currents in the spectral domain when we write routine for the computation of programs (programs) we define them in terms of:

$$M_{k\phi_S}(r_{wav.aperture},n,k) = \sum_{m=-n}^{n}(-j)^{|m|-|k|}\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}\sqrt{\frac{(n-|k|)!}{(n+|k|)!}}P^{n}_{|m|,sign(m)\cdot k}(\cos\theta_{12})M_{k\phi}(r_{wav.aperture},n,m), \quad (19)$$

$$M_{k\theta_S}(r_{wav.aperture},n,k) = \sum_{m=-n}^{n}(-j)^{|m|-|k|}\sqrt{\frac{(n+|m|)!}{(n-|m|)!}}\sqrt{\frac{(n-|k|)!}{(n+|k|)!}}P^{n}_{|m|,sign(m)\cdot k}(\cos\theta_{12})M_{k\theta}(r_{wav.aperture},n,m). \quad (20)$$

The base and test functions on the displaced waveguide are easily determined by using the relation (5), (11) and (12)

If the wavelength center has a f-coordinate different from zero, it is easy to express the vector-Legendre transformation of base / test functions using the following Fourier series rule $\widetilde{M}_{i1}(n,m) = \widetilde{M}_{i2}(n,m)e^{jm(\beta_1-\beta_2)}$ ie. it is possible to connect the vector-Legendre transformation of base / test function of different waveguides with the same θ- coordinates. koordinatama.

It is equally possible to determine the equivalent magnetic currents at the opening of each waveguide and calculate the mutual coupling of the aperture antenna on the spherical surface using equation (1).

3. Discussion of Numerical and Experimental Results

By using the above-described procedure, the calculation of the mutual admittance of the aperture antennas on the spherical surface of three different radii was made. The figures (Figures 2, 3, 4 and 5) also show the values of the mutual admittance from the literature ([3] and [4]). It is noticeable to match the results from the literature and the calculated results.

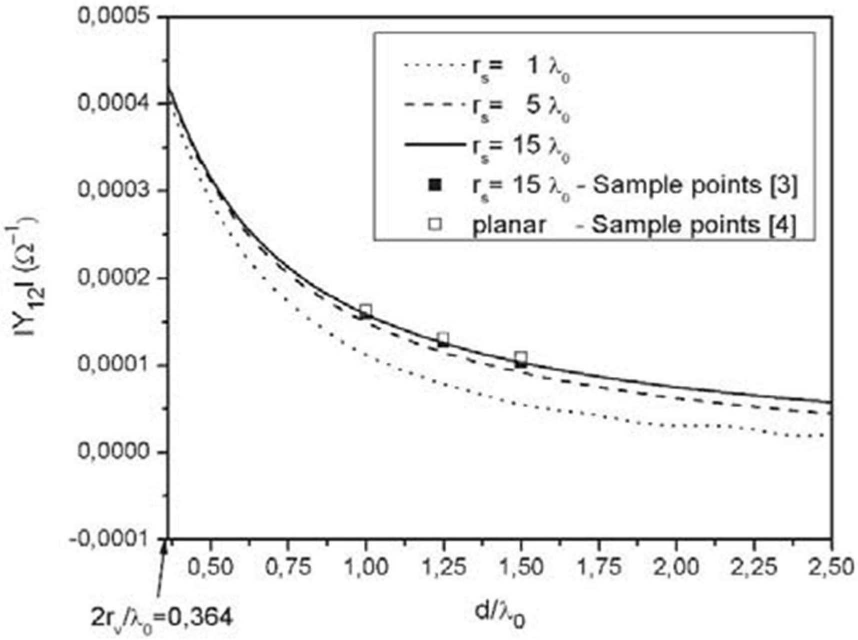

Figure 2. The mutual admittance of circular apertures on spherical surfaces for different radius of spherical surface – E plane

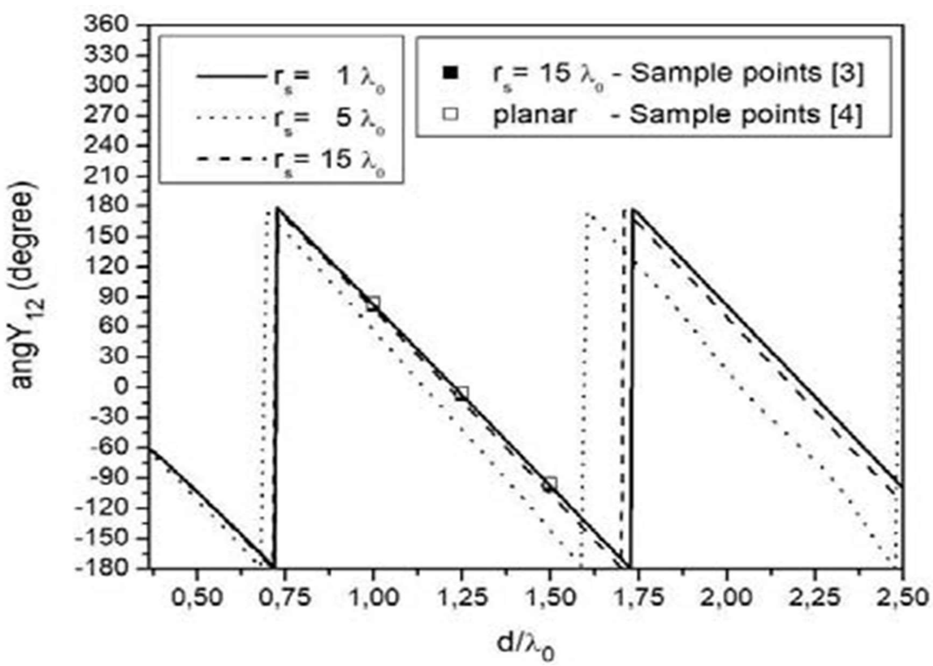

Figure 3. The mutual admittance angle of circular apertures on spherical surfaces for different radius of spherical surface – E plane

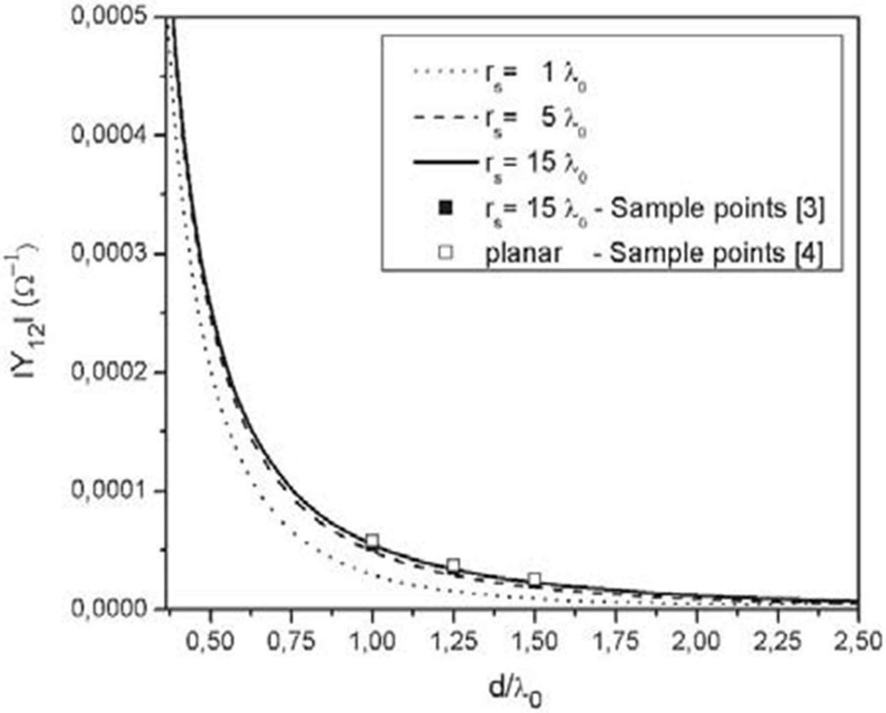

Figure 4. The mutual admittance of circular apertures on spherical surfaces versus distance between apertures centars for different radius of spherical surface – H plane

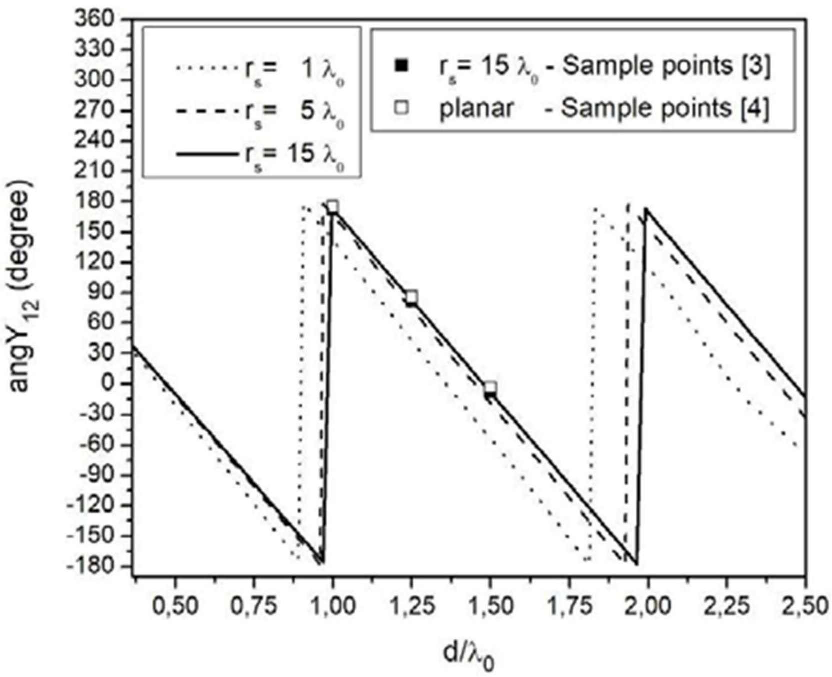

Figure 5. The mutual admittance angle of circular apertures on spherical surfaces for different radius of spherical surface – H plane

Verification of the procedure and its application to the radiation problem of the antenna aperture on the spherical surface is made with another comparison as shown in Figure 6. The radius of the circular apertures is 1.905 cm, and the distance between them is 6.35 cm.

As the reference curves, the curves for the planar case were used according to Lit 5. It can be seen that by increasing the spherical radius the results of the mutual coupling approach the planar results to Lit 5.

Furthermore, a model antenna array model was created on a aluminium spherical surface with radius of 30 cm radius. Radius of the circular waveguides are 6 cm (Figure 7.). The mutual coupling of the openings were measured, the first being positioned in the north pole (α_1=0 deg and β_1 = 0 deg) while the second in the position of α_2=56 deg and β_2 = 180 deg.Figure 8. shows an excellent match between measurement results and those obtained by simulation.

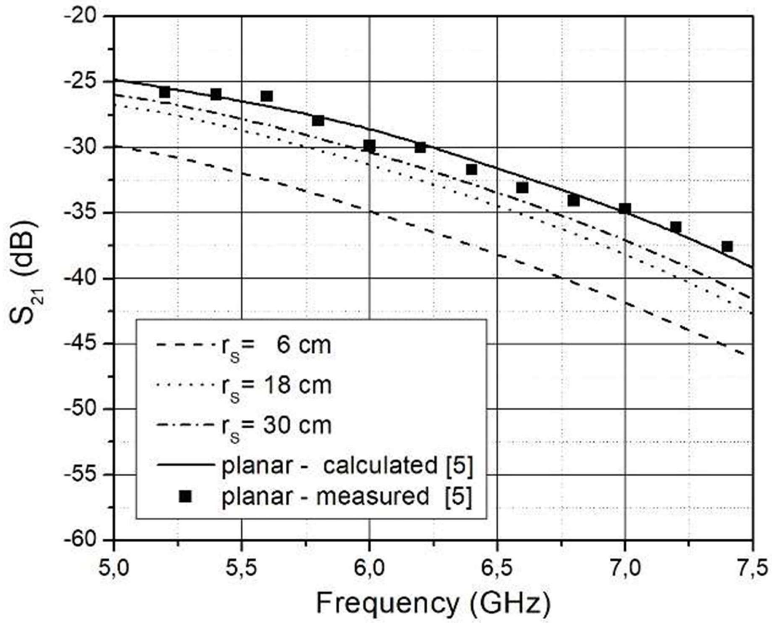

Figure 6. Calculated magnitude of S_{21} parameter of two circular apertures as a function of frequency for different radius of spherical surface and also calculated and measured magnitude of S_{21} parameter of two circular apertures on the planar surface – H plane [6]

Figure 7. Photo of the developed laboratory model of aperture antenna array: W1 - α_1= 0 deg and β_1 = 0 deg; W2 - α_2 = 56 deg and β_2 = 180 deg; r_w = 6cm; r_s = 30 cm

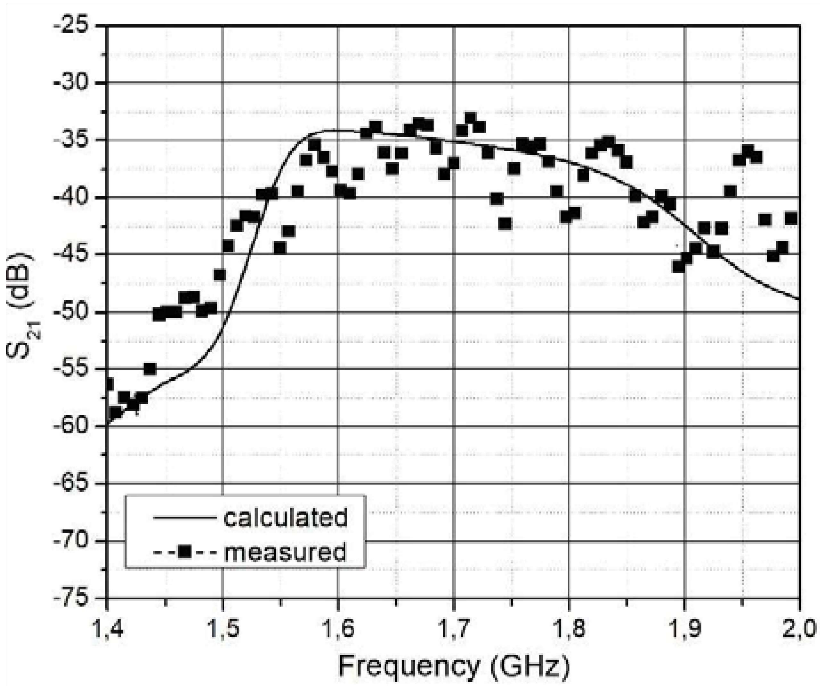

Figure 8. Calculated and measured magnitude of S_{21} parameter of two circular apretures (radius = 6cm) on spherical surfaces with radius = 30 cm - E plane [6]

4. Conclusion

The calculation of the mutual coupling (admittance) of the circular aperture of the antenna located on the spherical substrate using the moment method is considerably accelerated by using the Additional theorem for spherical harmonics. This Vilenkin's theorem was applied when calculating equivalent magnetic currents at the analyzed openings. Conversion of the coordinates and integration for returning to the spatial domain has been omitted, which is why the acceleration of this calculation procedure is compared with the classical procedure. The results of this calculation are compared with the results from available literature, and compared to the performed laboratory model. The results show excellent agreement with comparative results.

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