A Framework for the Digital Hilbert Transformer with Cascade Realization

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ABSTRACT: *In this work, we proposed a framework for the digital Hilbert transformer that can limit 90-degree deviations. We studied the cascade realization of the divisions of the structure to include phase sensitivity minimization of all-pass sections. The design is tested for efficiency, providing acceptable experimental results.*

Keywords: Digital Filters, Allpass Filters, Hilbert Transformers, Sensitivity Minimizations

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1. Introduction

Hilbert transformers (*HT*) are very important building blocks in both, analog and digital signal processing. They are used in telecommunications for generation of analytic and single-sideband signals [1] [2] and in many other modulation and demodulation schemes (mainly for splitting the narrowband signals to two (*I* and *Q*) components), in complex signals processing, in audio and video signal processing, and even in fields like mechanical vibration signal processing. Many approaches and methods of design of digital *HTs* have been developed in the last 50 years and most of them have been well systematized in [3]. The *FIR* based HTs are providing easily a linear phase response and unconditional stability but at the price of a very high transfer function (*TF*) order (say, several hundred), producing quite a high total delay and requiring higher power consumption. These disadvantages are eliminated in the *IIR* realizations, most often based on the usage of allpass structures. The theory of the allpass-based *HTs* is quite mature and several design methods using real or complex allpass structures have been summarized in [3]. Many new optimization-based methods for design of halfband filters and *HTs* have been proposed since then (including even frequency response masking technique [4]), but no specific methods for accuracy improvement have been reported. Meanwhile the practical importance of the *HTs* grew considerably with the extension of the frequency ranges and the growth of the proportion of the narrow-band signals, described as analytic, in telecommunications. The problem with the accuracy of the realization of the *HTs* is of paramount importance in many of these telecommunication applications, like in the maintenance of *I* and *Q* channels balance in a wide frequency range. When the *HTs* are realized using a fixed-point arithmetic (what is often the case in the portable and mobile communication equipment), the limited word-length may reduce considerably that accuracy and special measures have to be taken to prevent that. Higher accuracy could be achieved by designing the *HTs* with higher *TF* order, but the portability of the equipment is imposing another constraint – the power supply limitation. The main aim of this work is to try to improve the accuracy of the allpass-based *HTs* throughout minimization of their sensitivities. It will reduce the computational load and will permit shorter word-length and lower power consumption for given accuracy. The design procedures should be straightforward, without iterative and complicated optimization steps, in order to be easily used by practicing engineers and the structures have to be with the lowest possible *TF* order and complexity.

2. Design Procedure

An ideal Hilbert transformer (also known as a 90-degree phase shifter) is described in frequency domain as [5]

$$H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega < \pi \\ j, & -\pi \le \omega < 0 \end{cases} . \qquad (1)$$

A way to synthesize an *IIR* Hilbert transformer (called also a complex half-band filter) is to start with an odd-order halfband filter with specifications F_p, F_s, δ_p and δ_s, interconnected by the relations [3]

$$\delta_s = \sin(\Delta\varphi_{\max}/2);\ \delta_p = 1 - \sqrt{1-\delta_s^2};\ F_p = 0.5 - F_s\ ; \qquad (2)$$

and with a *TF* $G(z)$ that may be represented as a sum of two allpass *TFs* [3] [5]

$$G(z) = 0.5[A_1(z^2) + z^{-1}A_2(z^2)] . \qquad (3)$$

An “even-odd” decomposition (Figure 1) and the substitution

$$H(z) = 2G(-jz) \qquad (4)$$

must be applied in order to obtain the real allpass *TFs*. Thus

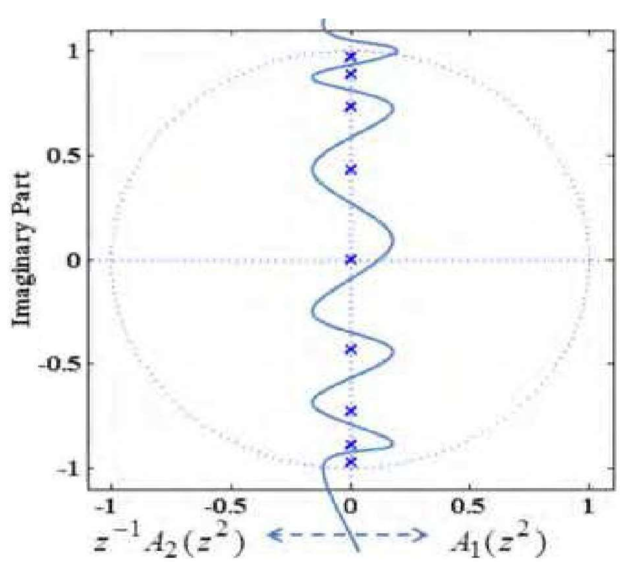


Figure 1. “Even-odd” decomposition of the *TF* poles

$$H_{HT}(z) = 2G(-jz) = [A_1(-z^2) + jz^{-1}A_2(-z^2)] \tag{5}$$

represents the *HT* as a complex sum of two real allpass functions, whose realization (for real input signal $x(n)$) is given in Figure 2. Details about the design are given in [3] [5].

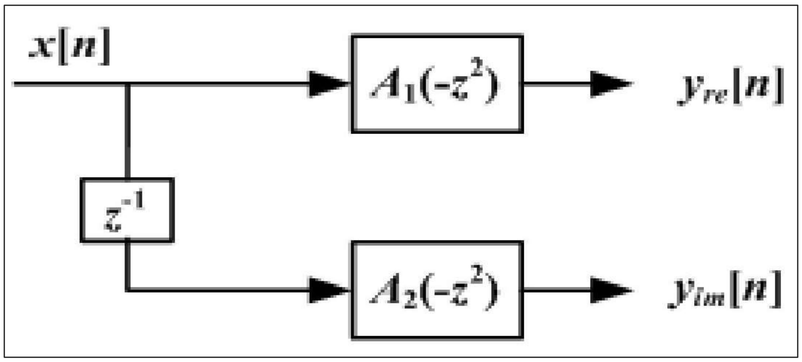


Figure 2. *HT* realization

3. Allpass Sections Realizations

The allpass *TFs* in Equation (3) are having all their poles on the imaginary axes, while those in Equation (5) are all on the real axes. In order to obtain higher accuracy in the 90° phase shifting in case of a limited word-length environment, the allpass TFs in Figure 2 could be realized as cascades of special second-order allpass sections. It follows from Fig. 1 that if a cascade realization would be used, as the possible real pole positions are scattered all around the real axes, the allpass sections with low sensitivities for all these positions will be needed.

We have studied [6] all known (about 20) first order sections and it was found that several low-sensitivity sections for every real pole position could be found. We select to use the most typical four of them, namely the *ST*1 section, providing low-sensitivity for poles near z=1, *MH*1 and *SC*, having low sensitivity for poles near $z = 0$ and *SV* section for poles near $z = -1$. The special sections are obtained from these real first order sections by changing the signs of the coefficients of the allpass TFs in Eq. (3) and by replacing z^{-1} by z^{-2} as it is shown in Figure 3. We denote these new second order allpass sections as *MH*1-2, *ST*1-2, *SV*-2 and *SC*-2.

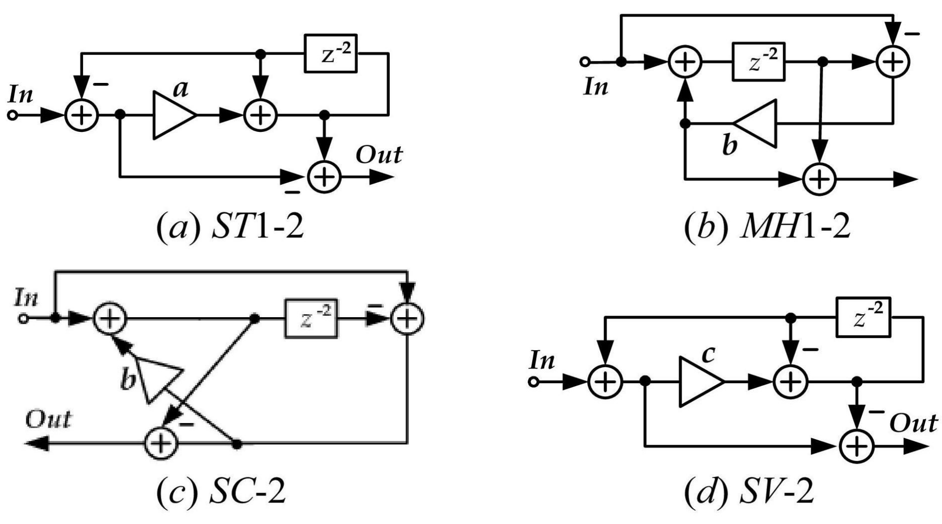


Figure 3. Different special second-order allpass sections

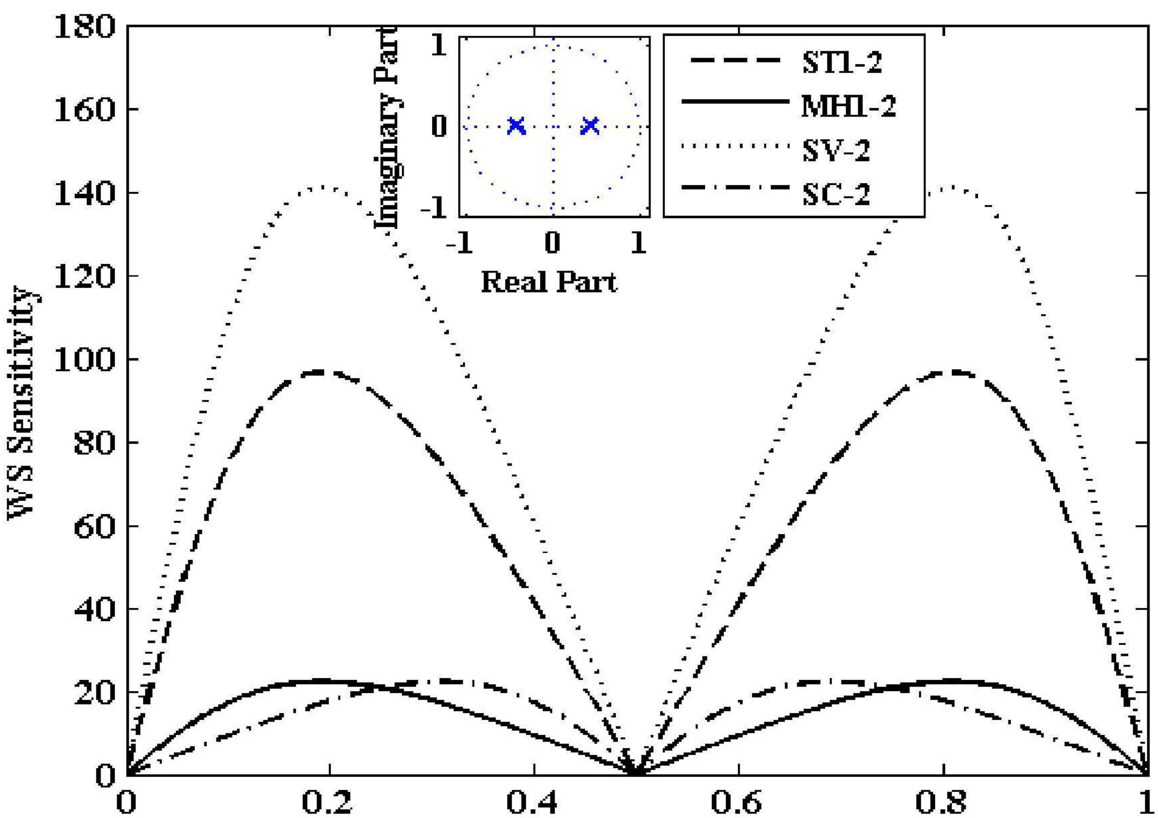
Their *TFs* are:

$$H_{ST1-2}(z)=\frac{-(1-a)+z^{-2}}{1-(1-a)z^{-2}}; \quad H_{MH1-2}(z)=\frac{-b+z^{-2}}{1-bz^{-2}}; \tag{6}$$

$$H_{SC-2}(z)=\frac{-b-z^{-2}}{1+bz^{-2}}; \qquad H_{SV-2}(z)=\frac{1-c+z^{-2}}{1+(1-c)z^{-2}}. \tag{7}$$

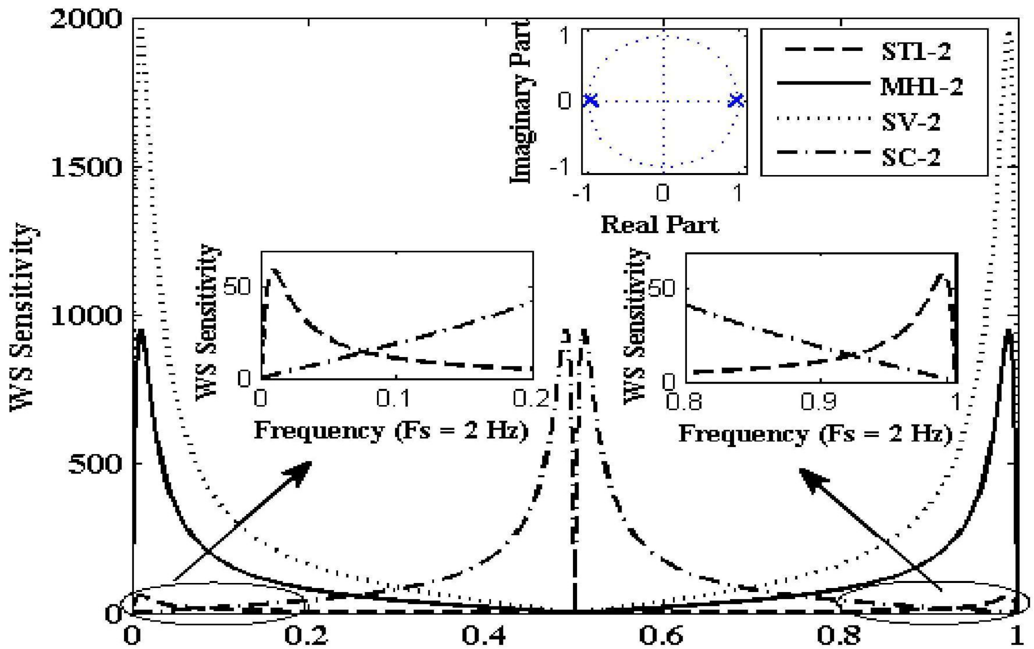
4. Allpass Sections Sensitivity Investigations

In Figure 4 *a*, *b* the worst-case (*WS*) phase-response sensitivities of the above mentioned four special sections are given for realizations with two different *TF* pole positions. The sensitivities are obtained by using the package *PANDA* [7]. By comparing the results with our previous investigations in [8] [9], it can be noted that the *WS* sensitivity behavior of the special second order sections is very similar to that of the corresponding first order sections but with the symmetry around the frequency $f = 0.5$. It is clearly seen that there exists a proper selection of the sections for every given *TF* pole position because of the significant difference between the maximal values of the sensitivities (in some cases it can reach more than 100 times especially for the poles near ±1).



Frequency (Fs = 2 Hz)

(a) $b_{MH1\text{-}2} = 0.18654$



Frequency (*Fs* = 2 *Hz*)

(*b*) $b_{MH1\text{-}2} = 0.94167$

Figure 4. Worst-case phase-sensitivities of second-order allpass sections (Figure 3) for two different *TF* poles positions

5. Overall Sensitivity Investigations

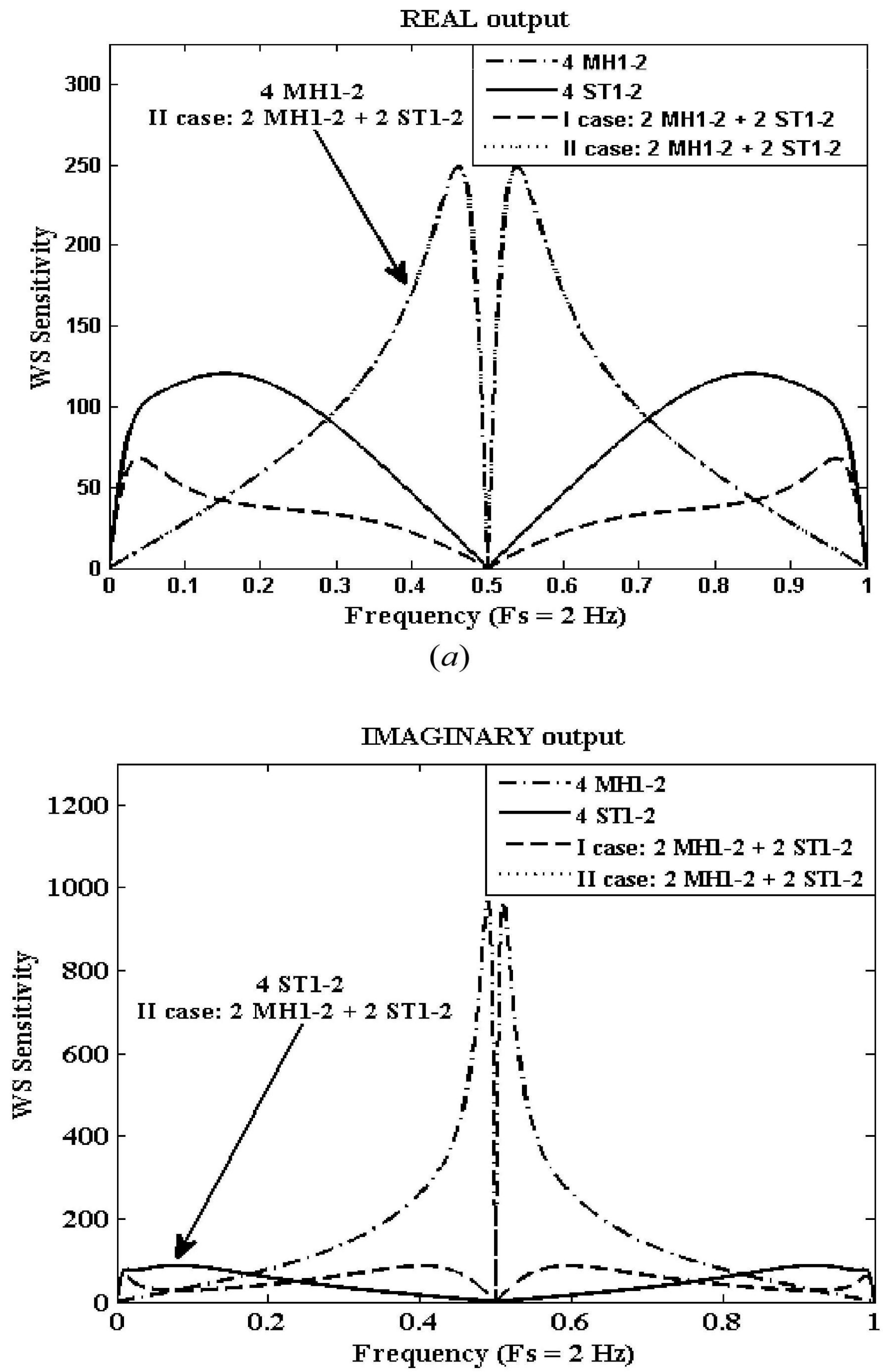
In order to estimate how the proper choice of the special sections will affect the behavior of the *HT* realization in a limited word-length environment, we have designed and investigated a ninth order HT having the TF poles positions given in Figure 1 (the initial elliptic half-band filter specifications are: passband frequency $F_p = 0.24$ and stopband attenuation $\delta_s = 0.01$ (R_s = 40 *dB*) $\Delta\varphi_{max} = 1.15°$).

Then, we have designed 4 different *HT* realizations (Figure 2). The first one was realized using the standard way (using only *MH*1-2 sections) and it is marked in the figures as "4*MH*1-2". The allpass sections selection for the other realizations is based on the sensitivity minimization of the individual sec- tions depending on their poles positions. Thus, in the second HT realization (denoted with "4 ST1-2") four *ST*1-2 sections were used. In the third and fourth implementations, two *MH*1- 2 and two *ST*1-2 sections have been selected. In the first case, we have a special section of each type in every branch of the realization, while in the second case – two *MH*1-2 sections are used in the upper branch (the real output) and two *ST*1-2 sections – for the imaginary output. The results for the overall sensitivity of the two branches are shown in Figure 5.

It appeared that the best configuration is with two MH1-2 and two ST1-2 sections, each in every branch (I case), providing the lowest overall sensitivity in both paths.

6. Investigation of the Influence of the Sections Combinations in the Branches

The phase difference between the two outputs in Figure 2 will not be exactly 90°. Over some frequency range (narrower han half of the sampling frequency) it will alternate around this value with amplitudes $\Delta\varphi_{max}$ depending ideally only on the selected value of δ_s Equation (2), but in reality – also on the design accuracy and on the parasitic effects of the digital realization.



(*a*)

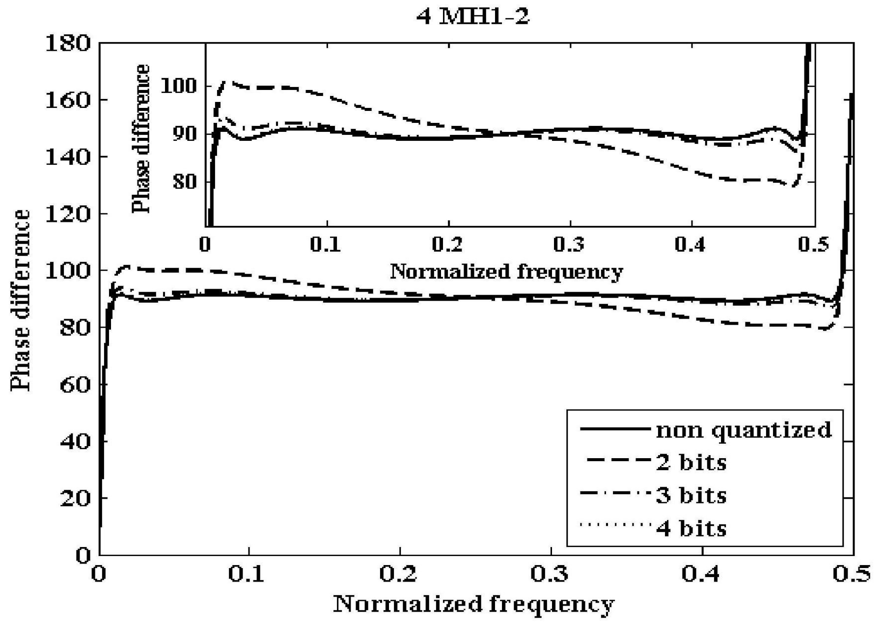
(*b*)

Figure 5. Worst-case phase-sensitivities of the *HT* (Figure 2) realized with different sets of allpass sections (for a 9-*th* order *HT*)

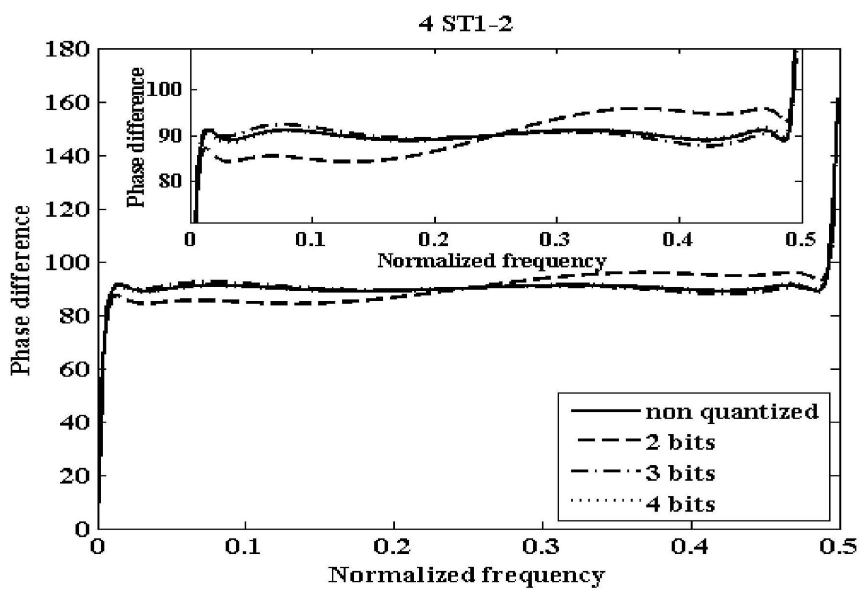
These additional deviations should be kept as lower as possible mainly by reducing the influence of the parasitic effects (by minimizing the sensitivities to the variations of the multiplier coefficients values). It will appear from what follows, that it might not be an easy straightforward procedure.

The accuracies of the HT realizations (the phase difference between the two branches) in a limited word-length environment are compared in Figure 6. Based on the results shown in Figure 5, it is natural to have a high sensitivity (to small changes in the two branches) of the phase difference between the two outputs in Figure 2 for 4 *MH*1-2 *HT* realization, but the results shown in Figure 6 *a* are quite surprising, compared to these in Figure 6 *b,c,d* (with minimized sensitivity). We suppose that this might be an effect due to some internal compensation between the parasitic effects in the branches, explained with the different signs of the sensitivities. The worst-case sensitivity *WS*, used in our investigations, is not able to reveal these mutual compensations, because it is eliminating the signs of the individual sensitivities.

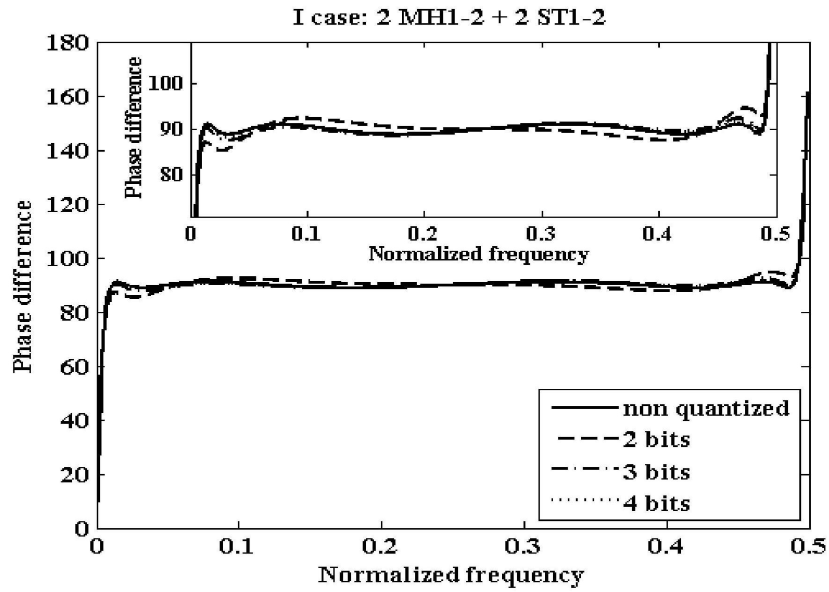
The highest accuracy, as it is shown in Figure 6, is achieved when we have two *MH*1-2 and two *ST*1-2 sections each inevery branch (I case) of the *HT*. In this case, the selection of the sections and their placement in the branches are made under the above mentioned observations.



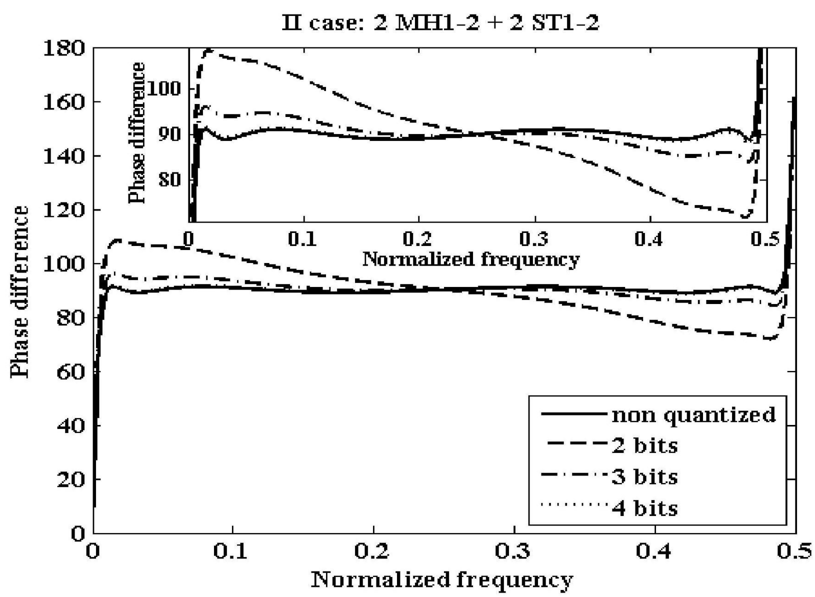
(*a*) using 4 *MH*1-2 sections



(*b*) using 4 *ST*1-2 sections



(*c*) using 2 *MH*1-2 and 2 *ST*1-2 sections - I case



(*d*) using 2 *MH*1-2 and 2 *ST*1-2 sections - II case

Figure 6. Word-length dependence of the accuracy of the *HT* phase difference for realizations with different allpass sections

As it can be seen after quantization to 2 bit (in *CSD* code) not only the fluctuations of the phase difference in Figure 6 *a,b,d* are growing very much above the ideal, but the range of frequencies over which this difference is approximately constant, is sharply reduced, while in Figure 6 *c* these parameters are practically unchanged.

The main conclusion of these investigations is that besides the sensitivity minimization, an additional step, consisting of a study of all possible combinations of the selected allpass sections within the branches, has to be introduced. A more general solution of this problem will be a derivation of a formula about the sensitivity of the phase quadrature to the changes of the multipliers' values, but it may appear to be a very difficult task.

7. Low-sensitivity Design Procedure

Taking into account all results so obtained, we propose the following design procedure:

1. Obtain *HHT* (z) Equation (5) by applying the standard design procedure from Sect. 2.

2. Decompose the TFs $A1(z^{-2})$ and $A2(z^{-2})$ to special secondorder allpass TFs and find where their poles are situated.

3. Select (from Figure 3) or develop new allpass sections realizing each couple of poles with the lowest sensitivity and verify this by sensitivity studies as these in Figure 4.

4. Investigate the overall sensitivities in the two branches of Figure 2 for all possible combinations of the selected allpass sections realizations in order to select the best set.

5. In case of a very high accuracy design, verify the selection by simulating the structure in a limited word-length environment (as in Figure 6).

We have applied this procedure for different sets of specifications and it was always possible to find an implementation clearly outperforming all the others as the case in Figure 6*c*.

8. Conclusion

A new approach to improve the accuracy of the allpass based Hilbert transformers (realized as two parallel branches) through sensitivity minimizations of each individual special second-order allpass section in the cascade realizations of the two branches was proposed in this paper. The design procedure is simple and straightforward, without iterative and complicated optimization steps and is achieving accuracy of realizations close to the ideal case (nonquantized coefficients). The low sensitivities so attained permit also a very short coefficients word-length, a higher processing speed and lower power consumption.

References

[1] Turner, C. S. (2009). An efficient analytic signal generator. *Signal Processing Magazine, 23*, 91-94.

[2] Rorabaugh, C. B. (2011). Notes on Digital signal processing: practical recipes for design, analysis, and implementation. *Notes 58-66, Prentice Hall.*

[3] Regalia, P. A. (1993). Special Filter Designs. In S. K. Mitra, J. F. Kaiser (Eds.), *Handbook for Digital Signal Processing* (pp. 909-931). John Wiley & Sons.

[4] Milic, L., Certic, J., Lutovac, M. (2010). A class of FRM-based allpass digital filters with applications in half-band filters and Hilbert transformers. In *Proceedings of the International Conference on Green Circuits and Systems (ICGCS)'2010*, Shanghai, China, 273-278.

[5] Mitra, S. (2006). *Digital signal processing: A computer-based approach*. McGraw-Hill.

[6] Stoyanov, G., Clausert, H. (1994). A comparative study of first-order digital all-pass structures. *Frequenz, 48*(9/10), 221-226.

[7] Sugino, H., Nishihara, A. (1990). Frequency-domain simulator of digital networks from the structural description. *Transactions of the IEICE of Japan, E73*(11), 1804-1806.

[8] Stoyanov, G., Nikolova, Z., Ivanova, K., Anzova, V. (2007). Design and realization of efficient IIR digital filter structures based on sensitivity minimizations. In *Proceedings of the 8th IEEE Conference TELSIKS'2007*, Nis, Serbia, 1, 299–308.

[9] Stoyanov, G., Nikolova, K., & Kawamata, M. (2011). Low-sensitivity design of allpass-based fractional delay digital filters. In F. P. Márquez (Ed.), *Digital filters* (pp. 155–178). Intech Publishing House.