Group Theory and Fourier Analysis of Finite Abelian Groups

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ABSTRACT: *Group characters are an essential concept in group theory and Fourier analysis of finite Abelian groups. For example, in many applications, such as spectral processing of binary or p-valued logic functions, it is often necessary to construct a table of groups of characters for a given group. This is a computationally demanding task, both from a space and time point of view, especially when working with large groups. The group characters are represented in matrix notation as rows of p × pm matrices, with p as cardinality and m as the number of variables in the set. GPUs, a highly parallel computing platform, can help solve this complex problem. In this paper, we will discuss how GPU processing can be used to construct tables of group characters for finite abelian groups, represented as direct products of cyclic subgroups of order p, and how it can be used to redistribute related computing tasks across GPU resources. The results of the experiments show that the proposed solution provides a significant speed-up compared to the C / C++ C character construction method executed on the CPU.*

Keywords: Abstract Harmonic Analysis, Finite Abelian Groups, Group Characters, Kronecker Product, GPU Computing, OpenCL

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1. Introduction

Abstract harmonic analysis is a mathematical discipline that evolved from the classical *Fourier analysis* by the replacement of the real group *R* with an arbitrary locally compact Abelian or compact non-Abelian group [3, 6, 9, 14, 15]. This implies the transition from the exponential functions, used in classical Fourier analysis and viewed as the group characters of *R*, to the *group characters*, in the case of Abelian groups, and the *group representations*, in the case of non-Abelian groups [6, 9].

Abstract harmonic analysis provides foundations for the formulation of many methods with significant applications in electrical engineering and computer science [9, 16, 17, 18, 19]. In these methods, it is often required to construct the group characters of various Abelian groups and use them in further computations. With that motivation, this paper presents a method for an efficient construction of group characters of finite Abelian groups using the graphics processing unit (GPU). This choice of hardware is made due to the fact that contemporary GPUs are highly parallel computing engines which can simultaneously serve as programmable graphics processors and scalable parallel computational platforms [1, 8, 13]. For a given group G, the construction of group characters can be expressed in terms of the Kronecker product of characters of its subgroups of smaller orders. In this formulation, the algorithm for the construction of group characters expresses a substantial inherent parallelism and, therefore, the GPU is a natural choice of hardware for the implementation of this algorithm. The experimental comparisons of the proposed implementation on the GPU and the C/C++ implementation of the same algorithm processed on the central processing unit (CPU) confirm this assumption.

The rest of the paper is organized as follows. The background theory is introduced in Section 2. In Section 3, we propose a mapping of the algorithm for the construction of group characters to the GPU and discuss the details of the respective programming implementation. The experiments are discussed in Section 4. We close the paper with Section 5, by presenting some conclusions and possible directions for further research.

2. Background Theory

In this section, we give a brief introduction to the theoretical background of the paper. For more detailed discussion of these topics, we recommend classical works such as [3, 15, 17], or more recent references [6, 9, 14].

We consider finite Abelian groups of the form $G = C^m_p = (\{0,1,..., p-1\}^m, \oplus_p)$, where C_p is the cyclic group of order p, and $\oplus_p$ is the componentwise addition modulo p.

The group characters $\chi^{(p)}_\omega(z)$, $z = 0, 1,..., p^{m-1}$, of the group G are defined as [9, 16, 17]:

$$\chi_\omega^{(p)}(z) = \exp\left(\frac{2\pi}{p} i \sum_{s=0}^{m-1} \omega_{m-1-s} z_s\right), \tag{1}$$

where $i = \sqrt{1}$, ω_s, $z_s \in \{0,1,..., p-1\}$, and

$$\omega = \sum_{s=0}^{m-1} \omega_s p^{m-1-s}, \quad z = \sum_{s=0}^{m-1} z_s p^{m-1-s} . \tag{2}$$

Example 1 The group character tables, for the cyclic groups Cp of orders $p = 2$, 3, and 4, are given in Table I, where $i = \sqrt{-1}$, $e_1 = -0.5\ (1- i\sqrt{3}) = \exp(2\pi i / 3)$, and $e_2 = e^*_1 = -0.5.(1+ i\sqrt{3}) = \exp(4\pi i / 3)$.

Cyclic group	C_2	C_3	C_4
Character table	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e_1 & e_2 \\ 1 & e_2 & e_1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

Table 1. Character Tables of Cyclic Groups

The group $G=C^m_p$ is the direct product of m elementary cyclic subgroups C_p. It follows, see for instance [3, 6, 15], that the character table of the group G is the Kronecker product of m character tables of its cyclic subgroup C_p.

Example 2 For the group C^2_3, the character table can be computed as the Kronecker product of two character tables of its cyclic subgroup C_3. In this way, only the character table of C_3 is computed through (1) and the character table for C^2_3 is generated as:

$$\left[C_3^2\right]=\left[C_3\right]\otimes\left[C_3\right]=\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}\otimes\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}=\begin{bmatrix}1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}\\1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&e_1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&e_2\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}\\1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&e_2\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}&e_1\cdot\begin{bmatrix}1&1&1\\1&e_1&e_2\\1&e_2&e_1\end{bmatrix}\end{bmatrix}\quad(3)$$

This property of the character table will be exploited in the mapping of the computation of the character table to the GPU.

3. GPU Construction Method

3.1. GPU Computing

The technique of performing general-purpose algorithms on graphics processors, known as GPGPU (*general-purposecomputing on GPUs*) or *GPU computing*, has recently become a subject of a fast growing research interest and practical application [1, 13].

This interest is mainly the result of two factors. First is the evolution of the *GPU* hardware towards a scalable, programmable, and highly parallel computing platform [1, 13], and the second is the development of the *Nvidia CUDA* [13] and *OpenCL* (*Open Computing Language*) [10] programming frameworks, based on the *C/C++* language,which made the immense *GPU* computational resources more accessible. For the implementation purposes, we use *OpenCL*, since it allows the development of the code that is both accelerated and portable across heterogeneous processing platforms (*GPUs, FPGAs, DSPs*) [8, 10].

3.2. Algorithm Mapping

The key task in porting algorithms to the *GPU* is their efficient mapping to the *SPMD* (*single program*, *multiple data*) *processing model* and the *multi-level memory hierarchy* of *GPUs* [1, 2, 8, 12, 13]. In the *GPU SPMD* model, a single data parallel function called a *kernel* is executed over a stream of data by many threads in parallel. A *thread* is the smallest execution entity and represents a single instance of the kernel. The execution of the kernel is controlled by the *host program* processed by the *CPU*.

The mapping of the algorithm for the construction of group character tables to the *GPU* is explained using Example 2.

The matrix $[C^2_3]$ in (3) has the following block structure:

$$\left[C_3^2\right]=\begin{bmatrix}B_{00}&B_{01}&B_{02}\\B_{10}&B_{11}&B_{12}\\B_{20}&B_{21}&B_{22}\end{bmatrix}\quad(4)$$

Blocks $B_{x,y}$ ($x, y = 0, 1, 2$) are the character tables for C_3 multiplied by the elements of the matrix $[C_3]$. Therefore, each block can be represented as:

$$B_{x,y}=c_{x,y}\cdot[C_3]=c_{x,y}\cdot\begin{bmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{bmatrix}=c_{x,y}\cdot\left[a_{i,j}\right],\quad(5)$$

where $c_{x,y}$, $a_{i,j} \in \{1, e_1,e_2\}$, $x, y, i, j = 0,1, 2$.

To each block we assign a thread $t = (x, y, a_{i,j})$, $x, y, i, j = 0, 1, 2$. Each thread performs a multiplication of $[C3]$ by a scalar, as in (5). Threads are organized into a two-dimensional (x, y) array corresponding to the matrices to be computed. Figure 1 represents the mapping of the character table computations to the GPU threads. Each thread processes a single block, which is indicated by a different color in Figure 1.

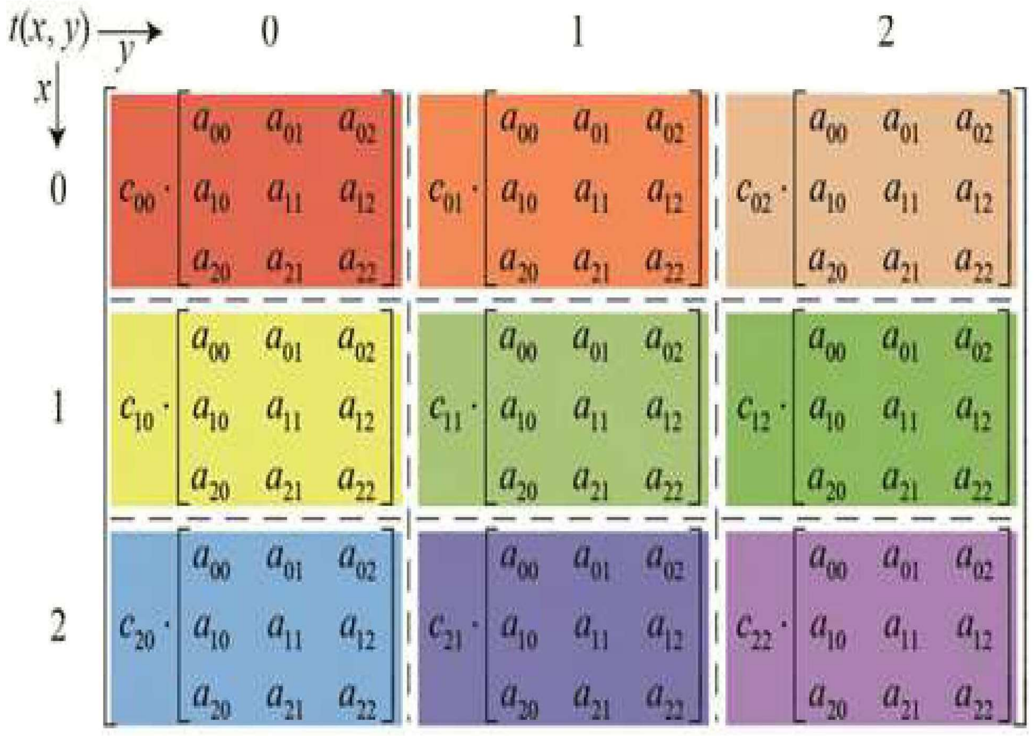


Figure 1. Mapping of the computations to the GPU threads for Example 2

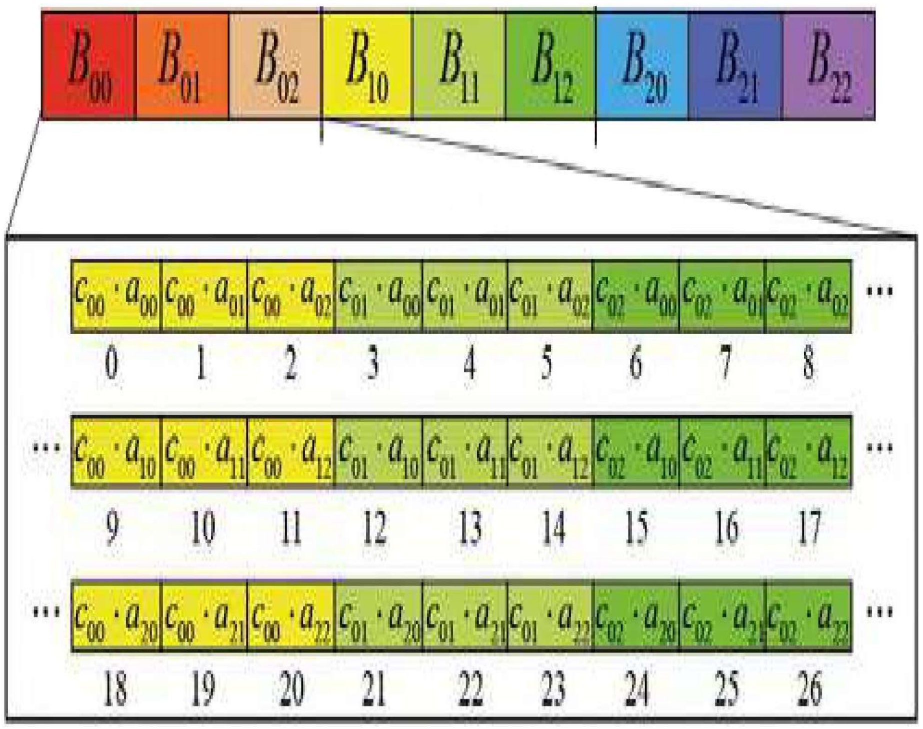
For the group C^2_3 in Example 2, we have nine threads in the first and only step of the algorithm (since this example involves only one Kronecker product), each performing the operation from (5) in parallel. In this case, indices of memory locations, where a thread $t(x, y, a_{i,j})$ stores the first element $(c_{x,y} \cdot a_{0,0})$ of the block, are computed as:

$$startElement \leftarrow x \cdot 3^3 + y \cdot 3 \qquad (6)$$

Indices of memory locations for the rest of the elements ($c_{x,y} \cdot a_{i,j}$, $x, y, i, j = 0, 1, 2$, except for the case $i = j = 0$) in a computed block are determined as:

$$nextElement \leftarrow startElement + i \cdot 3^2 + j \, . \qquad (7)$$

The results of the computations are stored in the *GPU* global memory which has a linear layout. Formulas for the computation of the memory location indices ((6) and (7)) lead to the GPU global memory access pattern which is, for Example 2, depicted in Figure 2. Coloring of the blocks and the memory locations in this figure corresponds to the thread coloring in Figure 1.



B_{00}
B_{01}
B_{02}
B_{10}
B_{11}
B_{12}
B_{20}
B_{21}
B_{22}
$c_{00}\cdot a_{00}$ $c_{00}\cdot a_{01}$ $c_{00}\cdot a_{02}$ $c_{01}\cdot a_{00}$ $c_{01}\cdot a_{01}$ $c_{01}\cdot a_{02}$ $c_{02}\cdot a_{00}$ $c_{02}\cdot a_{01}$ $c_{02}\cdot a_{02}$ ...
0 1 2 3 4 5 6 7 8
... $c_{00}\cdot a_{10}$ $c_{00}\cdot a_{11}$ $c_{00}\cdot a_{12}$ $c_{01}\cdot a_{10}$ $c_{01}\cdot a_{11}$ $c_{01}\cdot a_{12}$ $c_{02}\cdot a_{10}$ $c_{02}\cdot a_{11}$ $c_{02}\cdot a_{12}$ ...
9 10 11 12 13 14 15 16 17
... $c_{00}\cdot a_{20}$ $c_{00}\cdot a_{21}$ $c_{00}\cdot a_{22}$ $c_{01}\cdot a_{20}$ $c_{01}\cdot a_{21}$ $c_{01}\cdot a_{22}$ $c_{02}\cdot a_{20}$ $c_{02}\cdot a_{21}$ $c_{02}\cdot a_{22}$ ...
18 19 20 21 22 23 24 25 26

Structure of a global memory segment containing blocks

Figure 2. *GPU* global memory access pattern for Example 2.(4)Blocks

In the general case, in the k^{th} step of the algorithm, we perform the Kronecker product of a (p^k x p^k) matrix by the (p x p) matrix, and the result is a (p^{k+1} x p^{k+1}) matrix. Therefore, there are p^2 active threads in the first step of the algorithm, while in the k^{th} step, there are p^{2k} active threads. The index of the *GPU* memory location for the first entry ($c_{x,y}$. $a_{0,0}$) of the block is determined as:

$$startElement \leftarrow x \, . \, p^{k+2} + y \, . \, p, \tag{8}$$

The indices of the memory locations for the other elements ($c_{x,y}$. $a_{i,j}$, $i,j = 0, 1,\ldots, p$-1, except for the case $i = j = 0$)) in a block are:

$$nextElement \leftarrow startElement + i \, . \, p^{k+1} + j \, . \, p \, . \tag{9}$$

3.3. Features of the Mapping

The proposed method for computing the character tables has the following features:

1. The character table is stored as a vector of length p^{2m} obtained by the concatenation of rows of $[C^m_p]$. This allows reading the values of characters directly without any reordering.

2. Elements of $[C^m_p]$ computed by threads with the same first index and the successive second index are stored in neighboring memory locations. This automatically allows memory coalescing, due to which multiple data accesses to the *GPU* global memory are performed as a single memory transaction [2, 12].

3.4. Algorithm Implementation

A *GPGPU* program consists of two parts:

1. ***Host program***, which executes on the *CPU* and creates and controls the context for the execution of kernels as well as allocates and transfers data to the *GPU* memory.

2. ***Device program***, which is processed on the *GPU* and implements the *SPMD* kernels.

In the presented *OpenCL* implementation, the host program determines the character table for the cyclic subgroup C_p through (1). Notice that not all of the characters of C_p need to be computed by using (1), since, e.g., $e_{p-1} = e^*_i$ for $i = 1, 2, \ldots, [p/2]-1$. Thus, we compute half of the rows of the character table for C_p, while other rows are determined by using this property.

The host allocates *GPU* global memory space for two (p^m x p^m) matrices that are used as buffers to store the results of the application of the Kronecker product. This minimizes the communication between the host and the device, which is a bottleneck in the *GPU* computing [8, 12, 13]. Note that we have to reserve the space for (p^m x p^m) matrices at the beginning of the computation, since the size of the *GPU* buffers cannot be changed after their creation, otherwise, we would have to create buffers and transfer data between the host and the device for each step of the algorithm, as the resulting intermediate matrices increase in size. To minimize the memory bandwidth occupation on the *GPU* itself, we use the technique of buffer swapping [7]. For odd-numbered steps, the first matrix is used as the input to the kernel and the second matrix as the output. For even-numbered steps, the order is reversed.

The character table for C_p is stored in a (p x p) matrix and it is used as the second operand in the Kronecker product operation in each step. Since it is of a small size, we keep it in the constant *GPU* memory, which is cached. This allows much faster access and leads to improved program performance [12].

The Algorithm 1 presents a pseudo-code for the deviceprogram. Code in lines 2 and 6 implements (8) and (9), respectively. Since the characters of finite Abelian groups are complex numbers, elements of $[C^k_p]$, $[C_p]$, and $[C_p^{k+1}]$ are stored in the *GPU* buffers using the *float2 OpenCL* vector data type [10]. The first component in the vector variable stores the real part and the second component the imaginary part of the complex number.

Algorithm 1 Pseudo-code for the device program

```
1: $x, y \leftarrow$ acquire thread indices in the two-dimensional grid
2: $startElement \leftarrow x \cdot p^{k+2} + y \cdot p$
3: $adr1 \leftarrow x \cdot p^k + y$
4: for $i = 0$ to $p-1$ do
5:    for $j = 0$ to $p-1$ do
6:       $nextElement \leftarrow startElement + i \cdot p^{k+1} + j$
7:       $adr2 \leftarrow i \cdot p + j$
8:       $\left[C_p^{k+1}\right](nextElement).\mathrm{re} \leftarrow \left[C_p^k\right](adr1).\mathrm{re} \cdot \left[C_p\right](adr2).\mathrm{re} -$
                 $\left[C_p^k\right](adr1).\mathrm{im} \cdot \left[C_p\right](adr2).\mathrm{im}$
9:       $\left[C_p^{k+1}\right](nextElement).\mathrm{im} \leftarrow \left[C_p^k\right](adr1).\mathrm{re} \cdot \left[C_p\right](adr2).\mathrm{im} +$
                 $\left[C_p^k\right](adr1).\mathrm{im} \cdot \left[C_p\right](adr2).\mathrm{re}$
```

4. Experimental Results

The experiments reported in this section are performed using two hardware platforms, labeled **A** and **B**, respectively, and specified in Table 2. The GPU kernel performance analysis is done through the application of AMD APP Profiler 2.4 (for **A**) and Nvidia Parallel Nsigth 2.1 (for **B**), in accordance with instructions provided in [2, 12].

The referent *C/C++* implementation uses the *complex* data type from the Standard Template Library (*STL*) for the representation of the values of group characters. This data structure best corresponds to the *float2 OpenCL* vector data type [10] used for the same purpose in the *GPU* implementation. The referent *C/C++* implementation is compiled for the *x*64 platform using the *MS C++* compiler set to the maximum level of performance-oriented optimizations.

The results of the experiments performed on both test platforms for the construction of the character table for the groups C^3_m , m = 1, 2,…, 8, are resented in Figure 3. Notice that for p = 3 and m = 8, the size of the character table is p^m x p^m = 3^8 x 3^8 = 6561 x 6561, and, therefore, to complete the task, we have to compute and store 43 046 721 complex numbers. The OpenCL implementation processed on the GPUs outperforms the referent *CPU C/C++* implementation on both platforms and for all values of m used in the experiments. The speed-up is almost constant throughout the range for m, and it goes up to a factor of 7.8 × , on the test platform **A**, and up to a factor of 8.2 × on the platform **B**.

Platform	**A**	**B**
CPU	*AMD Phenom II N830 triple-core (2.1GHz)*	*Intel Core i7-920 quad-core (2.66GHz)*
RAM	*4GB DDR3 1066MHz*	*12GB DDR3-2000*
OS	*Windows 7 Ultimate (64-bit)*	
IDE	*MS Visual Studio 2010 Ultimate*	
SDK	*AMD APP 2.6*	*Nvidia GPU Computing 4.0*
GPU	*ATI Radeon 5650*	*Nvidia GTX 650 Ti*
engine speed	*650 MHz*	*900 MHz*
memory	*1 GB GDDR3 800 MHz*	*1 GB GDDR5 4.2 GHz*
processors	*80*	*384*

Table 2. Specification of Test Platforms

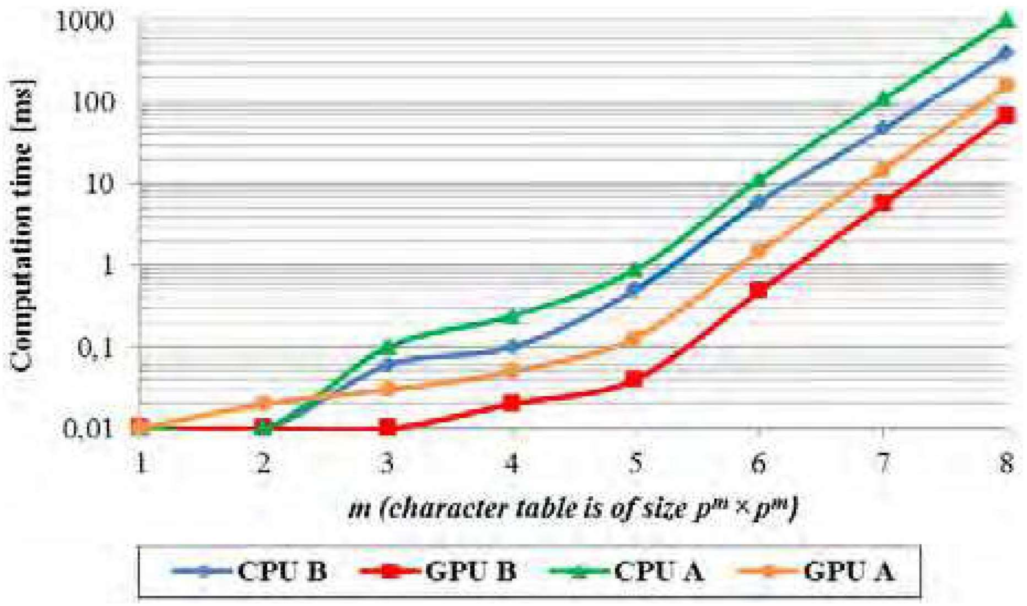


Figure 3. Computation times for the groups C^m_3 , m = 1, 2,…, 8, on *CPUs* and *GPUs* for the test platforms specified in Table 2

2. Conclusions

In this paper, we propose a method for the construction of characters of finite Abelian groups of the form $G = C^m_p = (\{0,1,..., p-1\}^m, \oplus_p)$, using the graphics processing unit (*GPU*) as the computational platform. We identify the sources of the parallelism available in the algorithm for construction of the character table for *G* formulated in terms of the Kronecker product. Based on this analysis, we devise a mapping of the computations to the *SPMD* processing model of the *GPU* and develop an *OpenCL* implementation of the algorithm. The experimental results obtained through the comparison of the proposed solution and the referent *C*/*C*++ implementation of the same algorithm show speed-ups of up to 7.8 × and 8.2 ×, depending on the platform, when using the *GPU* and, thus, confirm the validity of the proposed approach.

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