

# Solutions for Linear Inequality Systems

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**ABSTRACT:** The following are improved methods for finding the cut set Catalan values. These methods require calculating the number of specific solutions of a linear inequality system. The table of cut-set Catalan numbers is completed until  $n = 29$ .

**Keywords:** Cut Set Catalan Numbers, Dissection of a Convex Polygon, Triangulation

**Received:** 2 May 2023, Revised 20 July 2023, Accepted 1 August 2023

**DOI:** 10.6025/jdp/2023/13/4/111-114

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## 1. Introduction

Cut set Catalan numbers were introduced in [1] as a variation of the well-known classical Catalan numbers  $C_n$ . The Catalan number counts the number of dissections of a regular  $(n+2)$ -gon with labelled vertices by  $n-1$  non-intersecting diagonals into  $n$  triangles. The cut set Catalan numbers  $S_n$  are defined as follows: Let  $P_{n+2}$  be a regular convex  $(n+2)$ -gon with labelled vertices. Consider dissections of  $P_{n+2}$  by  $n-1$  non-intersecting diagonals into  $n$  triangles. Two such triangle sets  $T' = \{T'_1, \dots, T'_n\}$  and  $T'' = \{T''_1, \dots, T''_n\}$  are said to be isomorphic if there exists a bijection  $\phi: T' \rightarrow T''$  such that  $T'_i$  and  $T''_i$  are congruent for  $i=1, \dots, n$ . Then the cut set Catalan numbers  $S_n$  is defined to be the number of isomorphism classes of all such dissections of  $P_{n+2}$ .

In contrast to the classical Catalan numbers  $C_n$ , for  $S_n$  no explicit formula is known up to now. Some methods to determine  $S_n$  are presented in [1] and [2]. The new method in [4] is partly based on this results. We will now systemize the methods and analyze the ways to computing  $S_n$ . Moreover, the values of  $S_n$  are presented in [4] for  $n \leq 29$ .

The cut set Catalan numbers are registered in the On-Line Encyclopedia of Integer Sequences (OEIS, see [5]) with the sequence number A033961.

In [1] is proved that  $S_n$  equals the number of special solutions of a system of linear equalities. Then the method is improved in [2] and in [4]. Now we will present here these computer methods.

## 2. Special Linear System and $S_N$

A dissection  $\Delta$  of a convex  $(n+2)$ -gon  $P$  means here a dissection by  $n-1$  non-intersecting diagonals into  $n$  triangles. The label  $l_T(s)$  of a side  $s$  belonging to a triangle  $T$  in  $\Delta$  is defined as the number of vertices of  $P$  between the endvertices  $u$  and  $v$  of  $s$  on that part of the boundary of  $P$  not containing the third vertex  $w$  of  $T$ . Note that the label sum of the three sides belonging to  $T$  equals  $n-1$ . If  $a, b$  and  $c$  are the labels of the sides of a triangle  $T$  in  $\Delta$ , where  $a \leq b \leq c$ , then  $T$  is said to be of type  $(a, b)$ .

Let  $T_n$  be the set of the possible types of triangles:

$$T_n = \{(a, b) \in N_0 \times N_0; 0 \leq a \leq b \leq (n-1-a)/2\}. \quad (1)$$

We use  $\lambda_{a,b}$  for the number of triangles of type  $(a, b)$  in  $\Delta$ . It is easy to see that the upper bound  $\lambda_{a,b} \leq [(n+2)/(a+b+2)]$  holds for all  $(a, b) \in T_n$ .

A triangle of type  $(a, b)$  is called a boundary triangle if  $a=0$ , and an inner triangle if  $a \geq 1$ . The set of types for the inner triangles is

$$I_n = \{(a, b) \in N_0 \times N_0 \mid 1 \leq a \leq b \leq (n-1-a)/2\}. \quad (2)$$

A central triangle in  $\Delta$  is a triangle of type  $(a, b)$  where  $a+b \geq n/2-1$ . A central triangle can be only of certain types and the number of the central triangles in one dissection can be only one or two.

Since every diagonal  $d$  in  $\Delta$  is a side of two triangles, the sum of the two labels of  $d$  equals  $n$ . Thus, the number of occurrences of the label  $i$  in  $\Delta$  is equal to the number of occurrences of the label  $n-i$  in  $\Delta$  for  $i=1, \dots, n-1$ . These equalities and the requirement of the central triangle imply the following system of linear inequalities.

For  $(a, b) \in I_n, i=0, [(n-3)/2]$  let

$$\zeta_{a,b}(i) = \begin{cases} 0, & \text{if } a \leq i \text{ and } b > i \text{ or if } a+b \leq i-1, \\ 1, & \text{if } a \geq i+1, \\ -1, & \text{if } a \leq i, b \leq i, \text{ and } i \leq a+b < n-1-i, \\ -2, & \text{if } a \leq i, b \leq i, \text{ and } a+b \geq n-1-i. \end{cases} \quad (3)$$

Then the cut set Catalan number  $S_n$  equals the number of solutions in nonnegative integers of the following system of  $[(n+1)/2]$  linear inequalities with the  $[(n-1)^2+3]/12$  variables  $\lambda_{a,b} \leq [(n+2)/(a+b+2)], (a, b) \in I_n$ :

$$\begin{aligned} 0 &\leq 2 + \sum_{(a,b) \in I_n} \zeta_{a,b}(i) \lambda_{a,b} \quad \text{for } i = 0, \dots, [(n-3)/2], \\ 0 &\leq 2 - \sum_{\substack{(a,b) \in I_n \\ a+b=n/2-1}} \lambda_{a,b} - 2 \sum_{\substack{(a,b) \in I_n \\ a+b \geq n/2}} \lambda_{a,b} \quad \text{if } n \text{ is even,} \\ 0 &\leq 1 - \sum_{\substack{(a,b) \in I_n \\ a+b \geq (n-1)/2}} \lambda_{a,b} \quad \text{if } n \text{ is odd.} \end{aligned} \quad (4)$$

Thus, the values of  $S_n$  have been determined for  $n \leq 29$  without the need for examination of dissections of a polygon. The values for  $S_n$  known up to now are given in Table 1.

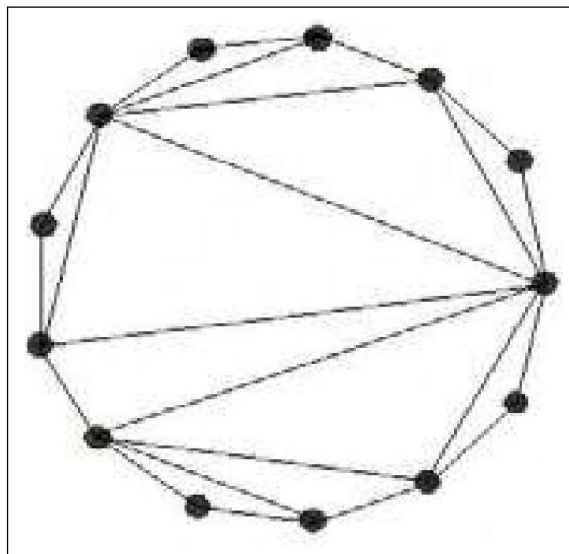


Figure 1. A dissection of 13-gon

n	2	3	4	5	6	7	8	9	10	11	12	13
$S_n$	1	1	2	2	4	6	11	17	35	57	115	203

N	14	15	16	17	18	19	20
$S_n$	412	745	1546	2838	5901	11154	23255

N	21	22	23	24	25	26
$S_n$	44263	93169	179214	377441	733151	1547068

n	27	28	29
$S_n$	3020878	6399874	12543862

Table 1. The values of  $S_N$

In the system of linear inequalities two inequalities, for  $i=0$  and  $i=1$ , can be deleted because they follow from the other inequalities. For example the system of linear inequalities for  $n=11$  gives the value  $S_{11}=57$ :

$$\begin{aligned}
 0 &\leq 2 - \lambda_{1,1} - \lambda_{1,2} - \lambda_{2,2} + \lambda_{3,3} \\
 0 &\leq 2 - \lambda_{1,2} - \lambda_{1,3} - \lambda_{2,2} - \lambda_{2,3} - \lambda_{3,3} \\
 0 &\leq 2 - \lambda_{1,3} - \lambda_{1,4} - \lambda_{2,2} - \lambda_{2,3} - 2\lambda_{2,4} - 2\lambda_{3,3} \\
 0 &\leq 1 - \lambda_{1,4} - \lambda_{2,3} - \lambda_{2,4} - \lambda_{3,3}
 \end{aligned} \tag{5}$$

### 3. The Methods for Computing $S_N$

The determination of values of  $S_n$  is doing with the aid of computer programs in C. The first program is already presented in [1] and it gives  $S_n$  for  $n \leq 19$ . It counts the number of solutions of a linear system of equalities. In [2] the method is improved

and we succeeded in evaluating  $S_n$  for  $20 \leq n \leq 25$ . This program counts the number of solutions of a linear system of inequalities from Equation (4) in nonnegative integers. A equivalent set of the system is constructed and it has a smaller number of inequalities. At this time the evaluation for  $S_{25}$  took about 100 hours. By the new values  $S_n$  for  $26 \leq n \leq 29$  the algorithm is once again improved [4]. The count the number of solutions we proceeded here in two steps.

**Step 1:** We first considered the subsystem of a system in Equation (4) consisting of the inequalities concerning  $i = [(n-1)/3], \dots, [(n-3)/2]$  and the last inequality. From  $i \geq [(n-1)/3]$  follow  $a \leq [(n-1)/3], a < i$  and  $\xi_{a,b}(i) \leq 0$ . We used a computer algebra system to determine all solutions of the subsystem in nonnegative integers. This step could be done in a few seconds.

**Step 2:** Using a simple algorithm, for each solution of the subsystem we counted the number of possibilities to choose the remaining variables  $\lambda_{a,b}$  such that the inequalities for  $i = 0, \dots, [(n-1)/3]-1$  and  $0 \leq \lambda_{a,b} \leq [(n+2)/(a+b+2)]$  are satisfied. For  $n=28$  and  $n=29$  the calculation were parallelized and distributed among more computers.

The program determines not only  $S_n$  but also all the relevant dissections of a polygon into triangles. The solutions of the linear system Equation (4) give the numbers of the inner triangles and for every dissection the numbers of boundary triangles are determined from a system of equalities, see [2]. How can be composed the dissection corresponding to this set of triangles? We start with the central triangle or two central triangles (possible for  $n$  even). For every diagonal we draw another triangle (every diagonal is side in two triangles and has two labels). This procedure ends successfully because the relations between the labels in the system of inequalities.

This is one of the solutions of a system from Equation (5):  $\lambda_{0,0} = 5, \lambda_{1,2} = \lambda_{0,1} = 2, \lambda_{1,4} = \lambda_{0,4} = 1, \lambda_{1,1} = \lambda_{1,3} = \lambda_{2,2} = \lambda_{2,3} = \lambda_{2,4} = \lambda_{3,3} = \lambda_{0,2} = \lambda_{0,3} = 0$ .

The corresponding dissection is given in Figure 1.

#### 4. Conclusion

At the time we work on the systems of inequalities for  $S_{30}$  and  $S_{31}$ . Based on a special classification of the dissections and the recursion formula from [2] we hope to get new values.

The cut set Catalan numbers are related to the other research areas: the maximum and minimum number of incongruent triangles in a dissection of regular convex polygon, see [3], the number of dissections by given number of triangle types, the numbers  $S_n(i)$  of cut set Catalan numbers with inner triangles etc.

#### Acknowledgement

This paper was supported by the project “Modern tendencies in the development of software technologies and algorithms, data processing and adequate training of specialists in these areas”.

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